ASYMPTOTIC EXPANSIONS OF THE SOLUTIONS FOR SMALL AND LARGE ROSSBY NUMBERS

by K. S. SINHA, *Department of Mathematics, D.A.V. College, Dehradun (U.P.)*

and

BHAGWAN SAHAI, *Department of Mathematics, D.B.S. College, Dehradun (U.P.)*

(*Received* 4 *October* 1976; *after revision* 12 *January* 1977)

Equations representing the motion of solids in incompressible, inviscid, rotating fluids are presented in non-dimensional form. Two iterative procedures for obtaining solutions to problems for small Rossby numbers $\left(R_0 = \frac{U}{\Omega a}\right)$ are given, and their merits and demerits are discussed. Using the technique of 'co-ordinate stretching', procedures for developing asymptotic expansions of the exact solutions for small and large Rossby numbers are worked out and the ranges of validity of these expansions are discussed. In some cases, the leading terms of the expansions satisfy linear differential equations which due to their similarity with Stoke's and Oseen's expansions of viscous flow theory have been called 'Stokes-like' and 'Oseen-like' expansions. The theory of asymptotic expansions is more or less satisfactory for small Rossby numbers, but for large Rossby numbers it presents some interesting problems which have been indicated; however, their solutions are not attempted.

INTRODUCTION

The motion of a body in a fluid which is inviscid and incompressible and which has a uniform angular velocity far away from the body is a subject of considerable interest. The various problems which have been worked out so far are based on one of the following two lines of approach. The first approach consists in assuming at the very outset a steady state motion. Thus, the possibility of posing initial value problems and the determination of their relationship with the steady state solutions is completely ruled out. In steady motion, by superposing a constant velocity on the whole field, the body is brought to rest and the motion of the fluid far away from the body is characterized by a uniform rotation Ω and a uniform translation U along the axis of rotation. The non-linear equations of motion with these conditions at infinity, in the case of axisymmetrical motion, reduce to one linear differential equation for Stoke's stream function Ψ (Long 1956, Nigam 1954).

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho} + \frac{4\Omega^2}{U^2}\Psi = -2\frac{\Omega^2 \rho^2}{U}$$

or

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial \rho} + 4\frac{\Omega^2}{U^2}\psi = 0 \qquad ...(1)$$

where

$$\Psi = -\frac{U}{2}\rho^2 + \psi.$$

This equation has been solved to discuss the motion past a sphere by Taylor (1922) and Long (1953), for motion past spheroids, disks and paraboloids of revolution, by Fadnis (1955a, b) and for motion created by sources and sinks at the axis by Long (1956). The same equation forms the basis of the investigations conducted by Fraenkel (1956) and Stewartson (1958).

In this paper, the linear approximations for solving problems of rotating fluids have been developed from the viewpoint adopted by Lagerstrom and Cole (1955) in respect of the solutions of the Navier Stokes equations and their "asymptotic expansions". Two approaches are presented, which, due to their similarity with the better known Stokes and Oseen linearization of the viscous flow theory, have been called here the "Stokes-like and Oseen-like" approximations. Defining Stokes- and Oseen-like variables in a slightly different manner, the ranges of validity of these approximations have been examined.

Equations of Motion and Linear Approximations

Consider the motion relative to the rotating frame (O, X, Y, Z) with OX always coinciding with OX' of the fixed frame (O, X', Y', Z') and rotating with a uniform angular velocity Ω about OX'. The velocity vector $\mathbf{V}'$ and $\mathbf{V}$ in the fixed and rotating frames are related by

$$\mathbf{V}' = \mathbf{V} + \Omega_\wedge r. \qquad ...(2)$$

The equations relative to the moving frame are

$$\nabla . \mathbf{V} = 0 \qquad ...(3)$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V} + 2\Omega_\wedge\mathbf{V} + \Omega_\wedge(\Omega_\wedge r) = -\frac{1}{\rho_1}\nabla p \qquad ...(4)$$

where p is the pressure and ρ_1, the density of the fluid. The last two terms of the left-hand side of (4) correspond to coriolis and centrifugal accelerations.

The problems of initial motion have been studied on the basis of the linearized equation.

$$\frac{\partial \mathbf{V}}{\partial t} + 2\Omega_\wedge\mathbf{V} + \Omega_\wedge(\Omega_\wedge r) = -\frac{1}{\rho_1}\nabla p \qquad ...(5)$$

taken in conjection with eqn. (3). This equation has been derived from eqn. (4) by neglecting the term $\mathbf{V}.(\nabla\mathbf{V})$ on the assumption that $\mathbf{V}$ is small everywhere in the fluid, such that its products with its derivatives are negligible throughout. Grace (1924, 26), Gortler (1944), Morgan (1951), Stewartson (1952, 1953a, 1953b, 1954) and Mallick (1957) used eqn. (5) in their investigations.

Squire (1956) linearizes eqn. (4) by neglecting $\mathbf{V}.(\nabla\mathbf{V})$ by $U\frac{\partial \mathbf{V}}{\partial x}$. Equation (4) then becomes

$$\frac{\partial \mathbf{V}}{\partial t}+U\frac{\partial \mathbf{V}}{\partial x}+2\Omega_{\wedge}\mathbf{V}+\Omega_{\wedge}(\Omega_{\wedge}r)=-\frac{1}{\rho_1}\nabla p. \qquad \ldots(6)$$

The term $U\frac{\partial \mathbf{V}}{\partial x}$ cannot be taken as a uniform approximation to $\mathbf{V}.(\nabla\mathbf{V})$ throughout the field. A thorough discussion of this point in connection with the relation between Oseen's and Stoke's approximation in a slow viscous flow (small Reynolds number) has been given by Lagerstrom and Cole (1955), Kaplun and Lagerstrom (1957) and Proudman and Pearson (1957).

In the description of any phenomenon determined jointly by the action of coriolis, local inertia, convective inertia and pressure forces, eqn (6) is superior to eqn. (5), since eqn. (5) neglects the interaction of the coriolis and convective inertia forces. The iterative procedures described below for eqns. (5) and (6) can be used to find higher order approximations. It also brings out the relation between the solutions of eqns. (5) and (6).

An Iterative Procedure for Small Rossby Numbers

Equations (3)–(6) may be expressed in non-dimensional form by making length, time and velocity dimensionless with the help of a characteristic length of the obstacle, Ω and U. The dimensionless variables are

$$r^*=\frac{r}{a},\ t^*=\Omega t,\ i=\frac{\vec{\Omega}}{\Omega},\ \mathbf{V}^*=\frac{\mathbf{V}}{U},\ \vec{\rho}^*=\frac{\rho}{\Omega U a},$$

$$P=\frac{p}{\rho_1}-\frac{\Omega^2}{2}(y^2+z^2),\ R_0=\frac{U}{\Omega a}\ \text{(Rossby number)} \qquad \ldots(7)$$

Equations (3) – (6) may be rewritten as

$$\nabla^*.\mathbf{V}^*=0 \qquad \ldots(8)$$

Inertia force $=\rho_1\frac{\partial \mathbf{V}}{\partial t}+\rho_1\mathbf{V}.(\nabla\mathbf{V})$

(local inertia force) (convective inertia force)

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + 2i_\wedge \mathbf{V}^* + \nabla^* P^* + R_0 \mathbf{V}^*.\nabla^* \mathbf{V}^* = 0 \qquad \text{...(9)}$$

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + 2i_\wedge \mathbf{V}^* + \nabla^* P^* = 0 \qquad \text{...(10)}$$

$$\frac{\partial \mathbf{V}^*}{\partial t^*} + 2i_\wedge \mathbf{V}^* + \nabla^* P^* + R_0 i.\nabla^* \mathbf{V}^* = 0. \qquad \text{...(11)}$$

The boundary conditions are

$\mathbf{V}^* = i$, at infinity,

and

$\mathbf{V}^*.n = 0$, on the surface.

The solutions of eqn. (9) can be obtained by iterating upon the solutions of eqn. (10) taken in conjunction with eqn. (8). The method consists in using the solutions of eqn. (10) to calculate the inertia term $\mathbf{V}^*.\nabla^*\mathbf{V}^*$ in eqn. (9) and solving the resulting equation for the next approximation. Since the boundary conditions at each stage of the iteration are independent of the Rossby number, this procedure is clearly equivalent to assuming an expansion of the flow in powers of the Rossby number. Another iterative method for obtaining the solutions of eqn. (8) can be developed by starting from eqns. (11) and (8). This would yield an expansion, each successive term of which would represent a uniformly valid higher approximation to the flow. In each step of the iteration, a lower order approximation would be used to calculate those particular inertia terms which are neglected in eqn. (11) and the resulting inhomogeneous form of the equation would be solved to the relevant degree of accuracy satisfying the given boundary conditions.

Coordinate Stretching

In the theory of rotating fluids, the range of applicability of the linear approximations of eqns. (5) and (6) are not clearly defined. Equation (4) indicates that any phenomenon in rotating fluids is governed by inertia, coriolis, centrifugal and pressure forces. It is quite possible that in any experiment some forces may be predominant over others. In such cases, the equations of motion can be considerably simplified by using 'boundary layer technique of coordinate stretching'. Theoretically, a number of possibilities present themselves; of these only a few interesting ones have been discussed here by expressing the equations of motion in non-dimensional form.

The equations for axisymmetric motion referring to the rotating frame in cylindrical coordinates (ρ, ϕ, x) in terms of Stoke's stream function ψ are

$$\frac{\partial D^2\psi}{\partial t} + 2\Omega \frac{\partial \chi}{\partial x} + 2\frac{\chi}{\rho^2}\frac{\partial \chi}{\partial x} + \frac{1}{\rho}\frac{\partial(\psi, D^2\psi)}{\partial(\rho, x)} + \frac{2}{\rho^2}\frac{\partial \psi}{\partial x} D^2\psi = 0 \qquad \text{...(12)}$$

$$\frac{\partial \chi}{\partial t} - 2\Omega \frac{\partial \psi}{\partial x} + \frac{1}{\rho} \frac{\partial(\psi, \chi)}{\partial(\rho, x)} = 0. \qquad ...(13)$$

The absolute velocity components are $(v_\rho, \Omega_\rho + v_\phi, v_x)$.

The components of the relative velocity are

$$v_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial x}, \; v_x = -\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}, \; v_\phi = \frac{\chi}{\rho},$$

and

$$D^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}.$$

These equations may by expressed in non-dimensional form by stretching the coordinates and introducing the following dimensionless variables :

$$\tilde{\rho} = R_0^n \frac{\rho}{a}, \; \tilde{x} = R_0^n \cdot \frac{x}{a}, \; \tilde{t} = R_0^\alpha \Omega t, \; \tilde{\psi} = R_0^{2n} \frac{\psi}{Ua^2}, \; \tilde{\chi} = R_0^n \frac{\chi}{Ua} \qquad ...(14)$$

where a is the characteristic length of the body; U, its velocity along OX; and Ω, the angular velocity of the undisturbed rotating fluid. A host of phenomena in rotating fluids can be discussed by taking different ranges of values of α and n. The following analysis is confined to $\alpha = 0$ only. Equations (12) and (13) expressed in non-dimensional form are

$$\frac{\partial D^2\tilde{\psi}}{\partial \tilde{t}} + 2\frac{\partial \tilde{\chi}}{\partial \tilde{x}} + R_0^{n+1}\left[2\frac{\tilde{\chi}}{\tilde{\rho}^2} \cdot \frac{\partial \tilde{\chi}}{\partial \tilde{x}} + \frac{1}{\tilde{\rho}}\frac{\partial(\tilde{\psi}, D^2\tilde{\psi})}{\partial(\tilde{\rho}, \tilde{x})} + \frac{2}{\tilde{\rho}^2}\frac{\partial \tilde{\psi}}{\partial \tilde{x}} D^2\tilde{\psi}\right] = 0 \qquad ...(15)$$

$$\frac{\partial \tilde{\chi}}{\partial \tilde{t}} - 2\frac{\partial \tilde{\psi}}{\partial \tilde{x}} + R_0^{n+1} \cdot \frac{1}{\rho}\frac{\partial(\tilde{\psi}, \tilde{\chi})}{\partial(\tilde{\rho}, \tilde{x})} = 0. \qquad ...(16)$$

The boundary conditions are

$$\left.\begin{array}{ll} \tilde{\psi} = 0, \text{ on } \tilde{x}^2 + \tilde{\rho}^2 = R_0^{2n} & \text{(normal velocity zero on the surface, taking sphere of radius } a \text{ as the body of revolution).} \\ \tilde{\psi} = \dfrac{R_0^{2n}}{2} \cdot \dfrac{\rho^2}{a^2} = \dfrac{\tilde{\rho}^2}{2} & \text{at infinity (uniform flow } U) \\ \tilde{\chi} = 0. & \text{at infinity (undisturbed rotation } \Omega) \end{array}\right\} \qquad ...(17)$$

Equations (15) and (16) give Stokes-like equations when $R_0 \to 0$ $(n + 1 > 0)$ and $R_0 \to \infty$ $(n + 1 < 0)$, provided $\tilde{\rho}$, $\tilde{x}$, $\tilde{t}$ are finite; when $n = 0$, the boundary conditions (17) and the variables $\tilde{\rho}, \tilde{x}$ become independent of R_0, while $\tilde{t}$ is always independent of R_0, since $\alpha = 0$. When $n = -1$, the equations become independent of R_0 (i.e., R_0 does not occur explicitly in the equations), but the boundary conditions involve R_0. This gives Oseen-like equations.

ASYMPTOTIC EXPANSIONS

The relation between the solutions of the linearized eqns. (5) and (6) and those of exact eqns. (12) and (13) can be best understood by invoking the ideas of asymptotic expansion. In fact, the exact solutions of (12) and (13) have asymptotic expansions of certain forms. These asymptotic expansions need not be necessarily uniformly valid for all values of ρ, x and $0 \leqslant t \leqslant \infty$. The expansions for $R_0 \to 0$ and $R_0 \to \infty$ are discussed below.

STOKES-LIKE EXPANSION

The proper Stokes-like variables correspond to $n = 0$ in (14), which may be defined as

$$\rho' = \frac{\rho}{a},\ x' = \frac{x}{a},\ t' = \Omega t,\ \psi_s' = \frac{\psi}{Ua^2},\ \chi_s' = \frac{\chi}{Ua}. \qquad \ldots(18)$$

Equations (12) and (13) reduce to

$$\frac{\partial D_s^2 \psi_s'}{\partial t'} + 2\frac{\partial \chi_s'}{\partial x'} + R_0\left[\frac{2\chi_s'}{\rho'^2}\cdot\frac{\partial \chi_s'}{\partial x'} + \frac{1}{\rho'}\frac{\partial(\psi_s', D_s^2 \psi_s')}{\partial(\rho', x')} + \frac{2}{\rho'^2}\frac{\partial \psi_s'}{\partial t'} D_s^2 \psi_s'\right] = 0 \qquad \ldots(19)$$

$$\frac{\partial \chi_s'}{\partial t'} - 2\frac{\partial \psi_s'}{\partial x'} + R_0\cdot\frac{1}{\rho'}\frac{\partial(\psi_s', \chi_s')}{\partial(\rho', x')} = 0. \qquad \ldots(20)$$

Taking the limit when $R_0 \to 0$, while ρ', x', t' remain finite, the Stokes-like limits,

$$\mathrm{Lim}_s\, \psi_s' = \psi_{s,0}' \text{ and } \mathrm{Lim}_s\, \chi_s' = \chi_{s,0}'$$

the above equations reduce to

$$\frac{\partial}{\partial t'} D_s^2 \psi_{s,0}' + 2\frac{\partial \chi_{s,0}'}{\partial x'} = 0 \qquad \ldots(21)$$

$$\frac{\partial}{\partial t'} \chi_{s,0}' - 2\frac{\partial \psi_{s,0}'}{\partial x'} = 0. \qquad \ldots(22)$$

The elimination of $\chi'_{s,0}$ from eqns. (21) and (23) gives

$$\frac{\partial^2}{\partial t'^2} D_s^2 \psi'_{s,0} + 4 \frac{\partial^2 \psi'_{s,0}}{\partial x'^2} = 0. \qquad \ldots(23)$$

This equation was obtained by Proudman and Pearson (1957) by eliminating p from (3) and (5). $\psi'_{s,0}$ is the first term in the asymptotic expansion of ψ'_s. The subsequent terms can be obtained by the repeated application of Stokes-like limit,

$$\psi'_{s,m+1} = \text{Lim}_s \frac{\psi'_s - \sum_{n=0}^{n=m} f_n \psi'_{s,n}}{f_{n+1}}. \qquad \ldots(24)$$

The expression for ψ'_s is

$$\psi'_s = \sum_{n=0}^{\infty} f_n(R_0)\, \psi'_{s,n}. \qquad \ldots(25)$$

Similarly,

$$\chi'_s = \sum_{n=0}^{\infty} f_n(R_0)\, \chi'_{s,n} \qquad \ldots(26)$$

without loss of generality the function $f_0(R_0)$ may be taken as unity and then

$$f_n(R_0) = \left(\frac{U}{\Omega a}\right)^n;$$

therefore

$$\frac{f_{n+1}}{f_n} \to 0 \quad \text{with} \quad R_0.$$

The Stokes-like limiting process, when $R_0 \to 0$, may be interpreted as the limiting process, when $\Omega \to \infty$ or $U \to 0$, so that ρ', x' and t' are finite. It can be seen from (18) that when $\Omega \to \infty$, ρ' and x' are finite at any fixed point of space, but t' becomes indefinitely large unless t is indefinitely small ($t \approx O(\Omega^{-1})$). Hence, the following conclusions can be drawn.

When $\Omega \to \infty$

(i) If the disturbance is three-dimensional and non-steady, the theoretical results will hold for all ρ and x for a very short time and cannot be extrapolated for large values of t.

(ii) If the disturbance tends to be steady, a two-dimensional regime independent of the x coordinate is possible.

When $U \to 0$

The results will be uniformly valid as the variables ρ', x' and t' are finite for finite values of ρ, x and t. Equations (19) and (20) are the proper equations for describing the flow in the vicinity of the body for large R_0 also. It is not possible to state *a priori* whether the asymptotic expansion corresponding to this limit will be uniformly valid at infinity or not. Dividing by R_0 and taking the limit when $R_0 \to \infty$ (interpreted as $U \to \infty$), while keeping ρ', x' and t' fixed, the equations reduce to

$$\frac{2\chi'_{s,0}}{\rho'^2} \cdot \frac{\partial \chi'_{s,0}}{\partial x'} + \frac{1}{\rho'} \frac{\partial(\psi'_{s,0}, D_s^2 \psi'_{s,0})}{\partial(\rho', x')} + \frac{2}{\rho'^2} \frac{\partial \psi'_{s,0}}{\partial x'} D_s^2 \psi'_{s,0} = 0 \quad \ldots(27)$$

$$\frac{1}{\rho'} \frac{\partial(\psi'_{s,0}, \chi'_{s,0})}{\partial(\rho', x')} = 0. \quad \ldots(28)$$

The solution of these equations gives the first term of the asymptotic expansion of the exact solution for $R_0 \to \infty$. This is the correct inner limit for this case.

The functions $\psi'_{s,0}$ and $\chi'_{s,0}$ will satisfy the boundary conditions at the surface and also at infinity.

Equations (27) and (28) can be further simplified as

$$\chi'_{s,0} = 4\psi'_{s,0} \quad \ldots(29)$$

$$(D^2 + 4)\, \psi'_{s,0} = -\, 2\rho^2. \quad \ldots(30)$$

The solutions of (30) satisfying the usual boundary conditions of the perfect fluid theory (relative normal velocity zero at the surface of the solid and the disturbance vanishing at infinity) will be indeterminate. It may be remarked that the time rate $\left(\frac{\partial}{\partial t}\right)$ terms do not appear in eqns. (27) and (28). It is not possible to find the manner in which t enters into $\psi'_{s,0}$ and $\chi'_{s,0}$. For these two reasons, the approach based on eqns. (27) and (28) for obtaining information about flow for $R_0 \to \infty$ does not seem to be fruitful.

When $R_0 \to \infty$ (interpreted as $\Omega \to 0$), while keeping ρ', x' and t' $\left(= \frac{Ut}{a}\right)$ fixed, eqns. (19) and (20) reduce to

$$\frac{\partial}{\partial t} D_s^2 \psi'_{s,0} + 2 \frac{\chi'_{s,0}}{\rho'^2} \cdot \frac{\partial \chi'_{s,0}}{\partial x'} + \frac{1}{\rho'} \frac{\partial(\psi'_{s,0}, D_s^2 \psi'_{s,0})}{\partial(\rho', x')}$$

$$+ \frac{2}{\rho'^2} \frac{\partial \psi'_{s,0}}{\partial x'} D_s^2 \psi'_{s,0} = 0 \quad \ldots(31)$$

$$\frac{\partial}{\partial t}\chi'_{s,0}+\frac{1}{\rho'^2}\frac{\partial(\psi'_{s,0},\chi'_{s,0})}{\partial(\rho',x')}=0. \qquad ...(32)$$

The asymptotic expansion corresponding to this limit may be defined as in (24). The non-linearity of eqns. (31) and (32) is a serious handicap in obtaining their solutions and hence will not be discussed here.

In general, the solutions of (23) will not exhibit any phenomenon depending on the interaction of the coriolis and convective forces. Higher order approximations will be required to explain those effects.

Oseen-Like Approximation

The Oseen-like non-dimensional variables correspond to $n = -1$ in (14) and may be defined as

$$\rho^+=\frac{\Omega\rho}{U},\ x^+=\frac{\Omega x}{U},\ \psi_0^+=\frac{\Omega^2}{U^3}\psi,\ \chi_0^+=\frac{\Omega\chi}{U^2},\ t^+=\Omega t. \qquad ...(33)$$

The relation between the Oseen-like and Stokes-like variables is

$$\rho^+=\frac{\rho'}{R_0},\ x^+=\frac{x'}{R_0},\ \psi_0^+=\frac{\psi'_s}{R_0^2},\ \chi_0^+=\frac{\chi'_s}{R_0},\ t^+=t'. \qquad ...(34)$$

Substitution of (33) in eqns. (12) and (13) gives

$$\frac{\partial}{\partial t^+}D_0^2\psi_0^+ + 2\frac{\partial\chi_0^+}{\partial x^+}+\frac{2\chi_0^+}{\rho^{+2}}\frac{\partial\chi_0^+}{\partial x^+}+\frac{1}{\rho^+}\frac{\partial(\psi_0^+, D_0^2\psi_0^+)}{\partial(\rho^+,x^+)}$$
$$+\frac{2}{\rho^{+2}}\frac{\partial\psi_0^+}{\partial x^+}D_0^2\psi_0^+=0 \qquad ...(35)$$

$$\frac{\partial\chi_0^+}{\partial t^+}-\frac{2\partial\psi_0^+}{\partial x^+}+\frac{1}{\rho^+}\frac{\partial(\psi_0^+,\chi_0^+)}{\partial(\rho^+,x^+)}=0. \qquad ...(36)$$

The boundary conditions on ψ_0^+ and χ_0^+ are

$$\left.\begin{aligned} &\psi_0^+=0 && \text{on the surface}\\ &\psi_0^+=-\frac{\rho^{+2}}{2} && \text{at infinity}\\ &\chi_0^+=0 && \text{at infinity.}\end{aligned}\right\} \qquad ...(37)$$

The first of these equations ensures that the normal velocity is zero on the body, while the remaining two equations give undisturbed flow at infinity. The asymptotic expansions of the solutions of eqns. (35) and (36) satisfying the boundary conditions (37) may be taken as

$$\psi_0^+ \sim \sum_{j=0}^{\infty} \psi_j^{++} (x^+, \rho^+, t^+, R_0) \qquad ...(38)$$

$$\chi_0^+ \sim \sum_{j=0}^{\infty} \chi_j^{++} (x^+, \rho^+, t^+, R_0) \qquad ...(39)$$

which are uniformly valid for $R_0 \to \infty$. To measure the degree of approximation, let us consider a sequence of function $f_j (R_0)$ such that

$$f_0(R_0) = 1, \quad \lim_{R_0 \to \infty} \frac{f_{j+1}(R_0)}{f_j(R_0)} = 0 \qquad ...(40)$$

and

$$\lim_{R_0 \to \infty} \frac{\psi_0^+ - \sum_{j=0}^{n} \psi_j^{++}}{f_n(R_0)} = 0$$

uniformly in space, i.e., the nth partial sum is valid to order f_n. The Oseen-like limit may be defined as

$$\text{Lim}_0\, F = \text{Lim}\, F \text{ as } R_0 \to \infty;\ x^+, \rho^+, t^+ \text{ fixed} \neq 0.$$

By repeated application of the Oseen-like limit with a suitably chosen sequence $f_j(R_0)$, the Oseen-like expansion of ψ_0^+ is

$$\psi_0^+ \sim \sum_{j=0}^{\infty} f_j(R_0)\, \psi_{0,j}^{++} (x^+, \rho^+, t^+) \qquad ...(41)$$

where

$$\psi_{0,0}^{++} = \text{Lim}_0\, \psi_0^+ \qquad ...(42)$$

and

$$\psi_{0,n+1}^{++} = \text{Lim}_0 \left[\frac{\psi_0^+ - \sum_{j=0}^{n} f_j(R_0)\, \psi_{0,j}^{++}}{f_{n+1}(R_0)} \right]. \qquad ...(43)$$

For finite bodies, $\psi_{0,0}^{++} = -\frac{1}{2}\rho^{+2}$ and $\psi_{0,1}^{++}$ then satisfies Oseen-like equations.

$$\frac{\partial}{\partial t^+} D_0^2 \psi_{0,1}^{++} + 2 \cdot \frac{\partial \chi_{0,1}^{++}}{\partial x^+} + \frac{1}{\rho^+} \frac{\partial \psi_{0,0}^{++}}{\partial \rho^+} \cdot \frac{\partial}{\partial x^+} D_0^2 \psi_{0,1}^{++} = 0. \qquad ...(44)$$

$$\frac{\partial}{\partial t^+}\chi^{++}_{0,1} - 2\frac{\partial \psi^{++}_{0,1}}{\partial x^+} + \frac{1}{\rho^+}\frac{\partial \psi^{++}_{0,0}}{\partial \rho^+}\cdot\frac{\partial \chi^{++}_{0,1}}{\partial x^+} = 0. \quad \text{...(45)}$$

Substituting the value of $\psi^{++}_{0,0}$ and eliminating $\chi^{++}_{0,1}$, the equations reduce to

$$\left(\frac{\partial}{\partial t^+} + \frac{\partial}{\partial x^+}\right)^2 D_0^2\,\psi^{++}_{0,1} + 4\frac{\partial^2\psi^{++}_{0,1}}{\partial x^{+2}} = 0. \quad \text{...(46)}$$

In general, this expansion is not uniformly valid near the body. This is, therefore, the proper outer limit. This fact makes it a singular perturbation problem. (Von Dyke 1964, Cole 1968, Eckhaus 1973 and O'Malley 1974).

The Oseen-like limiting process may be interpreted as the limiting process, when $\Omega \to 0$ or $U \to \infty$ or $a \to 0$, so that ρ^+, x^+ and t^+ are finite. When $\Omega \to 0$, ρ^+ and x^+ are finite for large ρ and x; t^+ is finite for large t. When $U \to \infty$, ρ^+ and x^+ are finite for large ρ and x, but t^+ is finite whenever t is finite. When $a \to 0$, ρ^+, x^+ and t^+ are finite for finite values of ρ, x and t. Hence, the following conclusions may be drawn.

$\Omega \to 0$

(i) If the disturbance is three-dimensional and non-steady, the theoretical results will hold for large values of t and will give information about regions far away from the body.

(ii) If the disturbance tends to a steady state, it will, in general, be three-dimensional, but may become two-dimensional in certain regions. Detailed solutions will be required for any information on this point. The results will hold only far large ρ, x and t.

$U \to \infty$

The same conclusions emerge as in (i) and (ii), except that the results are valid for large ρ and x but for all values of t.

$a \to 0$

The same conclusions emerge as in (i) and (ii), except that the results are valid for all ρ, x and t.

The above conclusions are subject to the limitation that the Oseen-like limit becomes singular on the surface of the solid.

Conclusion

The theory developed for motion in rotating fluids, when the Rossby number is small, appears to be quite satisfactory. The first iterative procedure in the first approximation totally neglects the interaction between the coriolis and the convective inertia forces. The effect of the neglected terms is taken care of in the subsequent iterations. The ultimate flow pattern about a sphere started from rest and made to move with a uniform velocity along the axis of the rotating fluid, according to this method, will have fore and aft symmetry. The second procedure is more suited to describe this situation, because it reveals an asymmetrical pattern which is well confirmed by the experiments of Long. This is due to the fact that it takes into account, at least partially, all the factors which go to determine the motion in rotating fluids.

The study of the asymptotic expansions of the exact solution gives some additional information about the range of applicability of the linear approximations. The first term of the asymptotic expansion of the stream function for small Rossby numbers satisfies a linear differential equation, which has been studied extensively by several workers. The Stokes-like approximation, when $U \to 0$ is uniformly valid, in the entire range of space and time variables. It seems quite adequate for $\Omega \to \infty$ also, when the disturbance tends to a steady state, but in the case of a non-steady disturbance, it fails to hold for any finite length of time. However, the Stokes-like limit for large Rossby numbers presents difficulties which appear to be almost insurmountable. An alternative method for studying the flow at large Rossby numbers is the Oseen-like limit. In this case, the results hold generally for regions far away from the body and have also restrictions on t. This limit is singular on the surface of the obstacle. The corresponding inner limit for this case is not known.

REFERENCES

Cole, J. D. (1968). Perturbation Methods in Applied Mathematics. Blaisdell Pub. Co., Massachusetts.

Eckhaus, Wiktor (1973). Matched Asymptotic Expansions and Singular Perturbations. North Holland Pub. Co., Amsterdam.

Fadnis, B. S. (1955a). Axisymmetric flow in perfect fluid—I. Motion about a spheroid and circular disc. *Bull. Calcutta math. Soc.*, **47**, 143-52.

———— (1955b). Axisymmetric flow in perfect fluid—II. *Bull. Calcutta math. Soc.*, **47**, 249-54.

Fraenkel, L. E. (1956). On the flow of rotating fluid past bodies in a pipe. *Proc. R. Soc.*, *A***233**, 506-26.

Gortler, H. (1944). Einge Bemer Kungen Ober Stromemgen in rotisrenden Flussigkeiten. *Z.A.M.M.*, **24**, 210.

Grace, S. F. (1924). A spherical source in a rotating liquid. *Proc. R. Soc.*, A**105**, 532-43.

———— (1926). On the motion of a sphere in a rotating liquid. *Proc. R. Soc.*, A**113**, 46.

Kaplun, S., and Lagerstrom, P. A. (1957). Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers. *J.M.A.M.*, **6**, 585-606.

Lagerstrom, P. A., and Cole, J. D. (1955). Examples illustrating expansion procedures for the Navier-Stokes equations. *J. Rat. Mech. Analysis.*, **4**, 819-81.

Long, R. R. (1953). Steady motion around a symmetrical obstacle moving along the axis of a rotating fluid. *Bull. Am. Met. Soc.*, **10**, 197-203.

Long, R. R. (1956). Sources and sinks at the axis of a rotating liquid. *Q. Jl Mech. appl. Math.*, **9**, 358-93.

Mallick, D. (1957). Motion induced by a sphere vibrating along the axis of rotation of an infinite rotating liquid. *Proc. natn. Inst. Sci., India.*, **23**, 544-59.

Morgan, G. W. (1951). A study of motions in a rotating liquid. *Proc. R. Soc.*, **A206**, 108-30.

Nigam, S. D. (1954). Hydrodynamical equations for the motion of bodies of revolution in non-viscous rotating liquid. *Q. Jl Mech. appl. Math.*, **7**, 458-61.

O'Malley, R. E. (1974). Introduction to Singular Perturbations. Academic Press, New York and London.

Proudman, I., and Pearson, J. R. A. (1957). Expansions at small Reynolds number for flow past a sphere and a circular cylinder. *J. Fluid Mech.*, **2**, 237-62.

Proudman, J. (1916). On the motion of solids in a liquid possessing vorticity. *Proc. R. Soc.*, **A92**, 408.

Squire, H. B. (1956). Rotating Liquids, Surveys in Mechanics (Ed. G. K. Batchelor). Cambridge University Press, London.

Stewartson, K. (1953a). A weak spherical source in a rotating liquid. *Q. Jl Mech. appl. Math.*, **6**, 45-49.

——————— (1953b). Slow motion of an ellipsoid in a rotating fluid. *Q. Jl Mech. appl. Math.*, **6**, 141-62.

——————— (1952). On the slow motion of a sphere along the axis of a rotating liquid. *Proc. Camb. phil. Soc.*, **48**, 168-72.

——————— (1954). Free motion of an ellipsoid in a rotating fluid. *Q. Jl Mech. appl. Math.*, **7**, 231-46.

——————— (1958). On the motion of a sphere along the axis of a rotating fluid. *Q. Jl Mech. appl. Math.*, **11**, 39-51.

Taylor, G. I. (1922). The motion of a sphere in a rotating liquid. *Proc. R. Soc.*, **A102**, 180-89.

Von Dyke, Milton (1964). Perturbation Methods in Fluid Mechanics. Academic Press, New York.