TWO DIAMETRAL CRACKS IN A CIRCULAR DISC*

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In this paper, we have solved the problem of determining the stress-intensity factors and the crack energy of two collinear radial cracks in a finite circular disc. By the use of Mellin transform technique, we reduce the problem to the solution of a singular integral equation. Approximate solution of the integral equation is obtained as a series of Chebyshev polynomials of the first kind. Expressions for the stress-intensity factors, the crack energy and the shape of the crack are derived in terms of the coefficients B_n of the series, which are determined from a system of linear algebraic equations. The numerical results for these quantities have been tabulated.

1. INTRODUCTION

In recent years, considerable attention has been devoted to the important problem of determining the stress in the vicinity of a crack due to its importance in the theory of brittle fracture (*see* Irwin 1958 and Barenblatt 1962). References on crack problems are available in Sneddon and Lowengrub (1969). The problem of a radial crack in a finite circular disc was solved recently by Tweed *et. al.* (1972). In this paper, we consider the problem of two symmetric cracks on the diameter of a finite circular disc. The problem is treated as that of plane stain. In Section 2, we give the general solution of the equilibrium equations by means of Mellin transforms. The problem is reduced to the solution of a singular equation in Section 3. In Section 4, approximate solution of the integral equation is assumed in terms of a series of Chebyshev polynomials of the first kind and algebraic equations for the determination of the coefficients B_n of the series are derived. In Section 5, we derive expressions, in terms of the coefficients B_n, for the shape of the deformed crack, the crack energy and the stress intensity factors at the edges of the crack. The numerical values of these quantities are given in the form of tables for the particular case of constant pressure on the crack.

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2. Solution of the Basic Equations

The two-dimensional plane strain equations of equilibrium are satisfied if the Airy's stress function χ satisfies the biharmonic equation

$$\nabla^4\chi(r, \theta) = 0 \qquad \text{...(1)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial\theta^2} \qquad \text{...(2)}$$

and the stresses and displacements are given in terms of χ by the following relations

$$\sigma_r = \frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\chi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)$$

$$\frac{\partial u}{\partial r} = \frac{1}{E_1}\left(\frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2} - \nu_1\frac{\partial^2\chi}{\partial r^2}\right)$$

$$\frac{\partial v}{\partial\theta} = \frac{1}{E_1}\left[r\frac{\partial^2\chi}{\partial r^2} - \nu_1\left(\frac{\partial\chi}{\partial r} + \frac{1}{r}\frac{\partial^2\chi}{\partial\theta^2}\right)\right] - u(r, \theta) \qquad \text{...(3)}$$

when r, θ denote the polar coordinates and

$$E_1 = \frac{E}{(1-\nu^2)}, \quad \nu_1 = \frac{\nu}{(1-\nu)} \qquad \text{...(4)}$$

E and ν being respectively the Young's modulus and the Poisson's ratio of the elastic material.

Applying the Mellin transform defined by

$$\bar{\chi}(p, \theta) = \int_0^\infty \chi(r, \theta)\, r^{p-1}\, dr \qquad \text{...(5)}$$

to eqn. (1), we find that it reduces to an ordinary differential equation

$$\left[\frac{d^4}{d\theta^4} + \{(p+2)^2 + p^2\}\frac{d^2}{d\theta^2} + p^2(p+2)^2\right]\bar{\chi}(p, \theta) = 0 \qquad \text{...(6)}$$

whose solution is

$$\bar{\chi}(p, \theta) = A\sin p\theta + B\cos p\theta + C\sin(p+2)\theta + D\cos(p+2)\theta \qquad \text{...(7)}$$

where A, B, C and D are arbitrary functions of p.

Also, from eqns. (3) and (5), we have

$$\{\overline{r^2\sigma_r};\ \overline{r^2\sigma_\theta};\ \overline{r^2\sigma_{r\theta}}\} = \left\{\frac{d^2\bar{\chi}}{d\theta^2} - p\bar{\chi};\ p(p+1)\,\bar{\chi};\ (p+1)\frac{d\bar{\chi}}{d\theta}\right\} \quad ...(8)$$

$$E_1(\overline{ru}) = \frac{p[1+\nu_1(1+p)]}{1+p}\,\bar{\chi} - \frac{1}{1+p}\frac{d^2\bar{\chi}}{d\theta^2} \quad ...(9)$$

$$E_1(\overline{rv}) = \frac{p^2(2+p)}{1+p}\int_\theta \bar{\chi}\,d\theta + \frac{1-\nu_1(1+p)}{1+p}\frac{d\bar{\chi}}{d\theta} + K(\theta), \quad ...(10)$$

where $K(\theta)$ is an arbitrary function of θ, which may be taken to be zero for the present problem.

3. Derivation of the Integral Equation

Consider a circular disc of unit radius containing a pair of symmetrical cracks, which are defined in plane polar coordinates (r, θ) by $\theta = 0, \pi, 0 < a \leqslant r \leqslant b < 1$. If the cracks are subjected to internal pressure $f(r)$, then we need to solve the following quarter-disc problem :

Problem 1 — Find the solution of the equations of elasticity for the quarter-disc $0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}$, satisfying the conditions :

(i) All stresses and displacements are bounded at the origin,

(ii) $\sigma_{r\theta}(r, 0) = 0,\ 0 \leqslant r < 1$,

(iii) $\sigma_{r\theta}\left(r, \frac{\pi}{2}\right) = v\left(r, \frac{\pi}{2}\right) = 0,\ 0 \leqslant r < 1$,

(iv) $\sigma_{r\theta}(1, \theta) = \sigma_r(1, \theta) = 0,\ 0 < \theta < \frac{\pi}{2}$,

(v) $\sigma_\theta(r, 0) = -\,p_0 f(r),\ a < r < b$,

(vi) $v(r, 0) = 0,\ 0 \leqslant r \leqslant a,\ b \leqslant r \leqslant 1$.

To find the solution of problem 1, we begin by superimposing the solutions of problems 2 and 3 below.

Problem 2 — This problem is to find a solution of the equations of elasticity for the quarter-disc $0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}$, which satisfies the conditions

(i) All the stressess and displacements are bounded at the origin,

(ii) $\sigma_{r\theta}(r, 0) = v(r, 0) = 0, \quad 0 \leqslant r \leqslant 1,$

(iii) $\sigma_{r\theta}\left(r, \frac{\pi}{2}\right) = v\left(r, \frac{\pi}{2}\right) = 0, \quad 0 \leqslant r \leqslant 1.$

Problem 3 — The problem here is to find a solution of the equations of elasticity for the quarter-plane $0 \leqslant r < \infty, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$, which satisfies the conditions

(i) All stresses and displacements are finite at the origin and tend to zero as $z \rightarrow \infty$,

(ii) $\sigma_{r\theta}(r, 0) = 0, \quad 0 \leqslant r < \infty,$

(iii) $\sigma_{r\theta}\left(r, \frac{\pi}{2}\right) = v\left(r, \frac{\pi}{2}\right) = 0, \quad 0 \leqslant r < \infty.$

Solution of Problem 3 — Utilizing eqns. (7), (8) and (10), we find that a stress function for this problem may be written as

$$\chi^{(3)}(r, \theta) = M^{-1}\left[\frac{A(p)}{(p+2)\sin\left(p\frac{\pi}{2}\right)}\left\{(p+2)\cos\left(\theta-\frac{\pi}{2}\right)p + p\cos\left(\theta-\frac{\pi}{2}\right)(p+2)\right\}; r\right] \quad \ldots(11)$$

where M^{-1} is the inverse Mellin transform and $-1 < \mathrm{Re}(p) < 0$.

From (8) - (11), we find that the stresses and displacements are now given by

$$r^2\sigma_r^{(3)}(r, \theta) = -M^{-1}\left[\frac{\phi(p)}{2\sin\left(p\frac{\pi}{2}\right)}\left\{(p+2)\cos\left(\theta-\frac{\pi}{2}\right)p + (p+4)\cos\left(\theta-\frac{\pi}{2}\right)(p+2)\right\}; r\right],$$

$$r^2\sigma_\theta^{(3)}(r, \theta) = M^{-1}\left[\frac{\phi(p)}{2\sin\left(p\frac{\pi}{2}\right)}\left\{(p+2)\cos\left(\theta-\frac{\pi}{2}\right)p + p\cos\left(\theta-\frac{\pi}{2}\right)(p+2)\right\}; r\right],$$

(continued on p. 347)

$$r^2\sigma_{r\theta}^{(3)}(r,\theta) = -M^{-1}\left[\frac{(p+2)\,\phi(p)}{2\sin\left(p\frac{\pi}{2}\right)}\left\{\sin\left(\theta-\frac{\pi}{2}\right)p + \sin\left(\theta-\frac{\pi}{2}\right)(p+2)\right\}; r\right],$$

$$ru^{(3)}(r,\theta) = \frac{1+\nu}{E}M^{-1}\left[\frac{\phi(p)}{2(p+1)\sin\left(p\frac{\pi}{2}\right)}\left\{(p+2)\cos\left(\theta-\frac{\pi}{2}\right)p + (p+4-4\nu)\cos\left(\theta-\frac{\pi}{2}\right)(p+2)\right\}; r\right],$$

$$rv^{(3)}(r,\theta) = \frac{1+\nu}{E}M^{-1}\left[\frac{\phi(p)}{2(p+1)\sin\left(p\frac{\pi}{2}\right)}\left\{(p+2)\sin\left(\theta-\frac{\pi}{2}\right)p + (p-2+4\nu)\sin\left(\theta-\frac{\pi}{2}\right)(p+2)\right\}; r\right] \quad \text{...(12)}$$

where the superscript (3) refers to the quantities for problem 3, and

$$\phi(p) = \frac{2p(p+1)\,A(p)}{(p+2)}. \quad \text{...(13)}$$

Solution of Problem 2 — It can easily be shown (*see* Timoshenko and Goodier 1951, Durelli *et al.* 1958) that a suitable choice of Airy stress function for this problem is

$$\chi^{(2)}(r,\theta) = \sum_{n=0}^{\infty}[a_{2n}r^{2n} + b_{2n}r^{2n+2}]\cos 2n\theta, \quad \text{...(14)}$$

and from (3), we now obtain

$$\sigma_r^{(2)}(r,\theta) = -\sum_{n=0}^{\infty}[2n(2n-1)\,a_{2n}r^{2n-2} + 2(2n+1)(n-1)\,b_{2n}r^{2n}]\cos 2n\theta,$$

$$\sigma_\theta^{(2)}(r,\theta) = \sum_{n=0}^{\infty}[2n(2n-1)\,a_{2n}r^{2n-2} + 2(2n+1)(n+1)\,b_{2n}r^{2n}]\cos 2n\theta,$$

(*continued on p.* 348)

$$\sigma_{r\theta}^{(2)}(r,\theta) = \sum_{n=1}^{\infty} [2n(2n-1)\, a_{2n} r^{2n-2} + 2n(2n+1)\, b_{2n} r^{2n}] \sin 2n\theta,$$

$$u^{(2)}(r,\theta) = -\frac{1+\nu}{E} \sum_{n=0}^{\infty} [2n a_{2n} r^{2n-1} + 2(n-1+2\nu)\, b_{2n} r^{2n+1}] \cos 2n\theta,$$

$$v^{(2)}(r,\theta) = \frac{1+\nu}{E} \sum_{n=1}^{\infty} [2n a_{2n} r^{2n-1} + 2(n+2-2\nu)\, b_{2n} r^{2n+1}] \sin 2n\theta \quad \ldots(15)$$

where the superscript (2) refers to solution of problem 2.

Solution of Problem 1 — By superimposing solutions of problems 2 and 3 above, we obtain

$$(\sigma_r, \sigma_\theta, \sigma_{r\theta}) = (\sigma_r^{(2)} + \sigma_r^{(3)}, \sigma_\theta^{(2)} + \sigma_\theta^{(3)}, \sigma_{r\theta}^{(2)} + \sigma_{r\theta}^{(3)}),$$
$$(u, v) = (u^{(2)} + u^{(3)}, v^{(2)} + v^{(3)}) \quad \ldots(16)$$

which is a solution of problem 1 satisfying conditions (i), (ii) and (iii) only. To solve problem 1 completely, we have to choose the function $\phi(p)$ and the sequences $\{a_n\}$ and $\{b_n\}$ in such a way that the conditions (iv) - (vi) are satisfied.

The boundary conditions (v) and (vi) will be satisfied if $\phi(p)$ satisfies the triple integral equations

$$M^{-1}[(1+p)^{-1}\phi(p); r] = 0, \quad 0 \leqslant r \leqslant a,$$
$$M^{-1}\left[\cot\left(p\frac{\pi}{2}\right)\phi(p); r\right] = -r^2[p_0 f(r) + F(r)], \quad a < r < b,$$
$$M^{-1}[(1+p)^{-1}\phi(p); r] = 0, \quad b \leqslant r \leqslant 1 \quad \ldots(17)$$

where $-1 < \mathrm{Re}(p) < 0$, and

$$F(r) = \sum_{n=0}^{\infty} [2n(2n-1)\, a_{2n} r^{2n-2} + 2(2n+1)(n+1)\, b_{2n} r^{2n}]. \quad \ldots(18)$$

If we assume

$$\phi(p) = \int_a^b \psi(t)\, t^{p+1}\, dt \quad \ldots(19)$$

where

$$\int_a^b \psi(t)\, dt = 0 \quad \ldots(20)$$

the first and third of eqns. (17) are satisfied identically and the second will be satisfied if $\psi(t)$ satisfies the integral equation

$$\frac{2}{\pi}\int_a^b \frac{t\psi(t)}{t^2 - r^2}\,dt = -p_0 f(r) - F(r), \quad a < r < b. \qquad \text{...(21)}$$

To evaluate $F(r)$ in terms of $\psi(t)$, we apply the remaining boundary conditions (iv), which yield the equations

$$\sum_{n=0}^{\infty} [2n(2n-1)\,a_{2n} + 2(2n+1)(n-1)\,b_{2n}]\cos(2n\theta)$$

$$= -M^{-1}\left[\frac{\phi(p)}{2\sin\left(p\frac{\pi}{2}\right)}\left\{(p+2)\cos\left(\theta-\frac{\pi}{2}\right)p + (p+4)\cos\left(\theta-\frac{\pi}{2}\right)(p+2)\right\};1\right],$$

$$\sum_{n=1}^{\infty} [2n(2n-1)\,a_{2n} + 2n(2n+1)\,b_{2n}]\sin(2n\theta)$$

$$= M^{-1}\left[\frac{(p+2)\,\phi(p)}{2\sin\left(p\frac{\pi}{2}\right)}\left\{\sin\left(\theta-\frac{\pi}{2}\right)p + \sin\left(\theta-\frac{\pi}{2}\right)(p+2)\right\};1\right]. \qquad \text{...(22)}$$

Solving eqns. (22) for a_{2n} and b_{2n}, we find that

$$b_0 = \frac{2}{\pi}M^{-1}\left[\frac{p^2+4p+2}{p(p+2)}\phi(p);1\right] \qquad \text{...(23)}$$

and that for $n \geqslant 1$

$$a_{2n} = -\frac{2}{\pi(2n-1)}M^{-1}\left[\frac{(p+2)(p^2-4n^2+4n-2)}{(p^2-4n^2)(p+2-2n)}\phi(p);1\right],$$

$$b_{2n} = \frac{2}{\pi(2n+1)}M^{-1}\left[\frac{(p+2)(p^2+2p+2-4n-4n^2)}{(p-2n)\{(p+2)^2-4n^2\}}\phi(p);1\right]. \qquad \text{...(24)}$$

Now substituting the values of b_0 and a_{2n}, b_{2n} $(n = 1, 2, \ldots)$ from (23) and (24) into (18), utilizing (19) and (20) and then working out the inverse Mellin transform, we obtain

$$F(r) = \frac{2}{\pi} \int_a^b g(r, t)\, \psi(t)\, dt, \quad a < r < b, \qquad \ldots(25)$$

where

$$g(r, t) = (t + t^{-1}) + t^3 [4(1 - r^2t^2)^{-3} - (1 - r^2t^2)^{-2} - (1 - r^2t^2)^{-1}] - t^{-1} [4(1 - r^2t^2)^{-3} - 3(1 - r^2t^2)^{-2}]. \qquad \ldots(26)$$

From (25) and (21), we now find that $\psi(t)$ is the solution of the integral equation

$$\int_a^b \frac{\psi(t)}{t - r}\, dt + \int_a^b k(r, t)\, \psi(t)\, dt = -\pi p_0 f(r), \quad a < r < b, \qquad \ldots(27)$$

where

$$k(r, t) = (t + r)^{-1} + 2g(r, t). \qquad \ldots(28)$$

4. Solution of the Integral Equation

To solve the integral eqn. (27), it would be convenient to change the variables r, t to ρ, τ respectively by the transformation :

$$t = \frac{a + b}{2} + \frac{b - a}{2}\tau, \quad r = \frac{a + b}{2} + \frac{b - a}{2}\rho, \qquad \ldots(29)$$

so that

$$\psi(t) = \psi\left(\frac{a + b}{2} + \frac{b - a}{2}\tau\right) = \Psi(\tau),$$

$$\frac{b - a}{2} k(r, t) = K(\rho, \tau),$$

$$-\pi f(r) = L(\rho), \qquad \ldots(30)$$

and the integral eqn. (27) becomes

$$\int_{-1}^{1} \frac{\Psi(\tau)}{\tau - \rho}\, d\tau + \int_{-1}^{1} K(\rho, \tau)\, \Psi(\tau)\, d\tau = p_0 L(\rho), \quad -1 < \rho < 1. \qquad \ldots(31)$$

Since the stress σ_θ should be infinite at the crack tips $r = a, b$ (i.e., $\rho = -1, 1$), $\Psi(\tau)$ must be infinite at these points [*see* eqn. (48)]. On the other hand, the displacement $v(r, 0)$ should be finite at the crack tips and hence $\Psi(\tau)$ is integrable at $\tau = 1, -1$ [*see* eqn. (43)].

With the above observations in mind, approximate solution of the singular integral eqn. (31) may be taken as (*see* Erdogan 1969) :

$$\Psi(\tau) \simeq p_0(1 - \tau^2)^{-1/2} \sum_{n=0}^{N} B_n T_n(\tau), \quad \text{...(32)}$$

where B_n's are constants and $T_n(\tau)$ are Chebyshev polynomials of the first kind. The condition (20) leads to

$$B_0 \equiv 0. \quad \text{...(33)}$$

Substituting for $\Psi(\tau)$ from (32) into the integral eqn. (31), we obtain

$$\sum_{n=1}^{N} \{B_n [\pi U_{n-1}(\rho) + g_n(\rho)]\} = L(\rho), \quad -1 < \rho < 1, \quad \text{...(34)}$$

where we have used

$$\int_{-1}^{1} T_n(\tau) (1 - \tau^2)^{-1/2} \frac{d\tau}{\tau - \rho} = \pi U_{n-1}(\rho), \quad n > 0, \quad \text{...(35)}$$

$$g_n(\rho) = \int_{-1}^{1} K(\rho, \tau) T_n(\tau) (1 - \tau^2)^{-1/2} d\tau, \quad n > 0 \quad \text{...(36)}$$

and $U_n(\rho)$ are Chebyshev polynomials of the second kind. Now, multiplying eqn. (34) by $U_j(\rho) (1 - \rho^2)^{-1/2}$ ($j = 0, 1, 2, \ldots N - 1$) and integrating with respect to ρ between -1 and 1, we obtain a set of algebraic equations for the determination of

$B_n (n = 1, 2, \ldots, N)$:

$$\frac{\pi^2}{2} B_{j+1} + \sum_{n=1}^{N} h_{nj} B_n = L_j, \quad j = 0, 1, 2, \ldots, N - 1 \quad \text{...(37)}$$

where we have used the orthogonality relation

$$\int_{-1}^{1} U_n(t) U_j(t) (1 - t^2)^{-1/2} dt = \begin{cases} 0, & n \neq j \\ \frac{\pi}{2}, & n = j > 0 \end{cases} \quad \text{...(38)}$$

and

$$h_{nj} = \int_{-1}^{1} g_n(t) U_j(t) (1 - t^2)^{-1/2} dt \quad \text{...(39)}$$

$$L_j = \int_{-1}^{1} L(t) U_j(t) (1 - t^2)^{-1/2} dt. \quad \text{...(40)}$$

5. Shape of the Crack, the Stress-Intensity Factors and the Crack Energy

For the shape of crack, we find, from (12), (15) and (16), that

$$v(r, 0) = -\frac{2}{E_1 r} M^{-1}\left[\frac{\phi(p)}{1+p}; r\right]. \quad \ldots(41)$$

Substituting for $\phi(p)$ from (19) into (41), we obtain

$$v(r, 0) = -\frac{2}{E_1 r}\begin{cases} 0, & r < t \\ \int_a^r r\psi(t)\, dt, & r > t. \end{cases} \quad \ldots(42)$$

Hence, the shape of the crack is given by

$$v(r, 0) = -\frac{2}{E_1}\int_a^r \psi(t)\, dt, \quad a < r < b$$

$$= -\frac{b-a}{E_1}\int_{-1}^{\rho} \Psi(\tau)\, d\tau, \quad -1 < \rho < 1. \quad \ldots(43)$$

The crack energy W is given by

$$W = p_0 \int_a^b f(r)\, v(r, 0)\, dr. \quad \ldots(44)$$

Substituting $v(r, 0)$ from (43), we find

$$W = -\frac{2p_0}{E_1}\int_a^b f(r)\left\{\int_a^r \psi(t)\, dt\right\} dr$$

$$= -\frac{2p_0}{E_1}\int_a^b \psi(t)\left\{\int_t^b f(r)\, dr\right\} dt.$$

If the crack is under constant pressure $p_0 f(r) = p_0$, then we have

$$W = \frac{2p_0}{E_1}\int_a^b t\psi(t)\, dt \quad \ldots(45)$$

where we have used the condition (20) for $\psi(t)$.

Substituting for t from (29) into (45) and noting that the condition (20) may be written as

$$\int_{-1}^{1} \Psi(\tau)\, d\tau = 0 \qquad \text{...(46)}$$

we find an alternative expression for the crack energy as

$$W = \frac{p_0(b-a)^2}{2E_1} \int_{-1}^{1} \tau\Psi(\tau)\, d\tau. \qquad \text{...(47)}$$

Now from (16), (15), (12) and (19), we find that

$$\sigma_\theta(r, 0) = F(r) + \frac{1}{r^2} \int_a^b M^{-1}\left[\cot p\pi : \frac{r}{t}\right] t\psi(t)\, dt$$

$$= F(r) + \frac{1}{\pi} \int_a^b \frac{\psi(t)\, dt}{t+r} + \frac{1}{\pi} \int_a^b \frac{\psi(t)\, dt}{t-r}$$

$$= F(r) + \frac{1}{\pi} \int_{-1}^{1} \frac{\Psi(\tau)\, d\tau}{\tau+\rho} + \frac{1}{\pi} \int_{-1}^{1} \frac{\Psi(\tau)\, d\tau}{\tau-\rho}, \quad \rho > 1,\ \rho < -1. \qquad \text{...(48)}$$

We observe that the singular part $\sigma_\theta^s(r, 0)$ of the stress $\sigma_\theta(r, 0)$ is contained only in the last term of the last line of eqn. (48). Substituting for $\Psi(\tau)$ from (32) into (48), we obtain

$$\sigma_\theta^s(r, 0) = p_0 \sum_{n=1}^{N} B_n T_n(\rho) \times \begin{cases} (\rho^2 - 1)^{-1/2}, & \rho < -1 \\ -(\rho^2 - 1)^{-1/2}, & \rho > 1 \end{cases} \qquad \text{...(49)}$$

where we have made use of the integral (*see* Gradshtein and Ryzhik 1965)

$$\int_{-1}^{1} \frac{P(\tau)\, d\tau}{(\tau-\rho)(1-\tau^2)^{1/2}} = \pi P(\rho) \begin{cases} (\rho^2 - 1)^{-1/2} & \rho < -1 \\ & + R(\rho) \\ -(\rho^2 - 1)^{-1/2} & \rho > 1 \end{cases} \qquad \text{...(50)}$$

when $P(\rho)$ is a polynomial in ρ and $R(\rho)$ denotes a regular function of ρ.

The stress-intensity factors at $r = a, b$, denoted respectively by K_a, K_b, are given by

$$\left.\begin{aligned} K_a &= \lim_{r \to a_-} [\{2(a - r)\}^{1/2} \sigma_\theta(r, 0)] \\ K_b &= \lim_{r \to b_+} [\{2(r - b)\}^{1/2} \sigma_\theta(r, 0)]. \end{aligned}\right\} \quad \ldots(51)$$

Substituting for r from (29) and for the singular part of $\sigma_\theta(r, 0)$ from (49) and noting that $r = a, b$ corresponds to $\rho = -1, +1$, we find that

$$\left.\begin{aligned} K_a &= p_0 \sqrt{\frac{l}{2}} \sum_{n=1}^{N} B_n(-1)^n \\ K_b &= p_0 \sqrt{\frac{l}{2}} \sum_{n=1}^{N} B_n \end{aligned}\right\} \quad \ldots(52)$$

where $l = b - a$ denotes the length of the crack.

Substituting for $\Psi(\tau)$ from (32) into (43) and (47) and noting that $T_n(t) = \cos n\theta$, when $t = \cos\theta$, we obtain the following expressions for the shape of the crack and the crack energy :

$$V(\rho) = E_1 v(r, 0) = p_0 l \sum_{n=1}^{N} \frac{B_n}{n} \sin(n \cos^{-1} \rho) \quad \ldots(53)$$

$$W = \frac{p_0^2 l^2}{E_1} \pi B_1. \quad \ldots(54)$$

TABLE I

Values of $10\, K_a/p_0$

$a \backslash b$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	2.272	3.294	4.121	4.828	5.431	5.924	6.300	6.590
0.2		2.246	3.199	3.938	4.548	5.046	5.433	5.755
0.3			2.239	3.171	3.877	4.439	4.875	5.223
0.4				2.236	3.158	3.840	4.359	4.746
0.5					2.235	3.147	3.799	4.255
0.6						2.233	3.129	3.710
0.7							2.229	3.066
0.8								2.208

When the crack is opened by a constant internal pressure (i.e., $p_0 f(r) = p_0$, a constant), eqns. (37) have been solved numerically for $N = 20$, by evaluating h_{nj} and L_j numerically from (39), (40) and (36). The numerical values of K_a, K_b and W are given in Tables I – III for all possible combinations of values of $a = 0.1, 0.2, \ldots, 0.8$ and $b = 0.2, 0.3, \ldots, 0.9$ $(b > a)$. The numerical values of $V(\rho)$ for $\rho = -0.75$, -0.50, -0.25, 0.00, 0.25, 0.50, 0.75 are given in Table IV for all possible combinations of values of $a = 0.1, 0.3, 0.5, 0.7$ and $b = 0.2, 0.3, \ldots, 0.9$ $(b > a)$. It may be noticed that $V(\rho) \equiv 0$ for $\rho = -1, 1$ for all possible values of a, b $(b > a)$.

TABLE II

Values of $10\, K_b/p_0$

$a \backslash b$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	2.261	3.237	4.000	4.656	5.266	5.906	6.725	8.066
0.2		2.244	3.189	3.926	4.570	5.206	5.986	7.310
0.3			2.239	3.174	3.906	4.572	5.333	6.633
0.4				2.237	3.170	3.915	4.677	5.935
0.5					2.236	3.174	3.980	5.177
0.6						2.237	3.202	4.336
0.7							2.241	3.389
0.8								2.284

TABLE III

Values of W/W_0, $W_0 = \pi p_0^2 l^2/E_1$

$a \backslash b$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.013	1.031	1.043	1.051	1.053	1.049	1.038	0.995
0.2		1.004	1.010	1.014	1.015	1.012	1.005	0.971
0.3			1.001	1.003	1.003	1.001	0.995	0.968
0.4				1.000	1.000	0.998	0.994	0.973
0.5					1.000	0.998	0.995	0.981
0.6						0.999	0.998	0.997
0.7							0.999	0.998
0.8								0.999

TABLE IV

Values of 10 $V(\rho)/p_0$

a	$\rho \backslash b$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	− 0.75	0.672	1.373	2.098	2.829	3.546	4.219	4.797	5.121
	− 0.50	0.879	1.793	2.730	3.671	4.590	5.454	6.195	6.585
	− 0.25	0.982	1.999	3.038	4.077	5.094	6.055	6.885	7.287
0.1	0.00	1.013	2.060	3.125	4.190	5.236	6.238	7.130	7.580
	0.25	0.981	1.991	3.016	4.042	5.058	6.052	6.987	7.568
	0.50	0.877	1.777	2.691	3.608	4.525	5.449	6.383	7.128
	0.75	0.670	1.356	2.051	2.752	3.463	4.209	5.051	6.058
	− 0.75			0.662	1.327	1.986	2.626	3.210	3.627
	− 0.50			0.867	1.737	2.602	3.443	4.216	4.757
	− 0.25			0.970	1.942	2.911	3.857	4.737	5.340
0.3	0.00			1.001	2.006	3.008	3.996	4.934	5.607
	0.25			0.970	1.942	2.916	3.885	4.837	5.612
	0.50			0.867	1.738	2.611	3.493	4.398	5.271
	0.75			0.662	1.327	1.998	2.685	3.440	4.405
	− 0.75					0.661	1.317	1.947	2.464
	− 0.50					0.866	1.726	2.557	3.241
	− 0.25					0.968	1.931	2.868	3.651
0.5	0.00					1.000	1.996	2.977	3.832
	0.25					0.968	1.935	2.900	3.812
	0.50					0.866	1.733	2.615	3.542
	0.75					0.662	1.326	2.017	2.880
	− 0.75							0.660	1.279
	− 0.50							0.864	1.683
	− 0.25							0.967	1.893
0.7	0.00							0.999	1.974
	0.25							0.968	1.938
	0.50							0.867	1.764
	0.75							0.662	1.381

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