EFFECT OF SECONDARY TERMS ON BENDING OF BEAMS

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The bending of a uniformly loaded rectangular beam clamped at both the ends is considered by taking into account some higher order terms in displacement components. The numerical results obtained for transverse displacement and stresses are compared with those of Timoshenko and classical theories.

1. INTRODUCTION

The classical theory of bending of beams has been of great interest in the application to engineering. But due to its oversimplified assumptions, that the cross-sections of the deformed beam remain plane and normal to the neutral surface, the theory is justified only for thin beams. Timoshenko (1921) modified this theory by assuming that sections remain plane after bending but do not remain normal to the neutral surface. This amounts to include a shear deformation which remains constant throughout the thickness of the beam. But as the shear cannot be constant throughout the thickness, Timoshenko introduced a constant in shear force to average it out over the thickness. Timoshenko theory has received a great deal of attention, specially in vibration problems. Later on, Goodier (1938) tackled the problem of bending of beams and plates by starting with the elasticity equations. Recently, Solar (1968) has investigated the problem of bending of thick rectangular beams by reducing the plane elasticity equations to a set of infinite coupled ordinary differential equations by expressing all dependent variables in series of Legendre polynomials in the thickness coordinates. A proper truncation of the series gives a set of finite equations for approximate theories. Solar and Tsai (1970) has also investigated the problem of locally loaded plane orthotropic beams on similar lines.

In the present paper, instead of starting with stress equations of elasticity, we have started with an assumption for displacement components. The displacement components are assumed to be infinite series in thickness coordinates and only a few terms are retained for our purpose. This gives rise to a cubical variation of normal stress and parabolic variation of shear stress, along the thickness. The equations of equilibrium are derived by the use of Hamilton's energy principle. They are solved for a beam clamped at both the ends. The numerical results computed for transverse displacement and normal and shear stresses are compared with Timoshenko and classical theories. It is found that the transverse displacement in our theory is larger than Timoshenko and classical theories and the difference is very much appreciable. In our case shear stress comes out to be maximum at the neutral surface and zero at the upper and lower facings. The normal stress along the thickness varies somewhat in a similar manner as given by Solar.

2. Displacements, Strains and Stresses

Let us consider a rectangular beam of length a, breadth b and depth h. Taking the middle point of left end as origin, the beam is referred to the Cartesian coordinates x, y, z. These coordinates are taken in the directions of length, breadth and depth respectively. The middle plane of the beam is taken as $z = 0$. Let p be the load per unit area applied on the upper facing of the beam.

The displacement components u, v, w in the directions of x, y, z respectively, are taken to be

$$\left.\begin{aligned} u(x, z) &= zu_1(x) + (z^3/h^2)\, u_3(x) + (z^5/h^4)\, u_5(x) + \ldots \\ v(x, z) &= 0 \\ w(x, z) &= w_0(x) + (z^2/h^2)\, w_2(x) + (z^4/h^4)\, w_4(x) + \ldots . \end{aligned}\right\} \quad \ldots(2.1)$$

We have expanded u in odd powers and w in even powers of z as u is expected to be skew-symmetric and w to be symmetric with respect to the neutral surface.

Retaining only first two terms in the expansions of u and w and neglecting the remaining terms, the strain components are given by

$$\left.\begin{aligned} \epsilon_x &= zu_{1,x} + (z^3/h^2)\, u_{3,x},\ \epsilon_y = 0,\ \epsilon_z = (2z/h^2)\, w_2 \\ \epsilon_{xy} &= 0,\ \epsilon_{yz} = 0,\ \epsilon_{zx} = u_1 + w_{0,x} + (z^2/h^2)(3u_3 + w_{2,x}) \end{aligned}\right] \quad \ldots(2.2)$$

where a comma followed by a suffix denotes the differentiation with respect to that suffix.

Using the relation (2.2) and the stress-strain relations

$$\left.\begin{aligned} \sigma_x &= (\lambda + 2\mu)\, \epsilon_x + \lambda(\epsilon_y + \epsilon_z),\ \sigma_{xy} = \mu\epsilon_{xy} \\ \sigma_y &= (\lambda + 2\mu)\, \epsilon_y + \lambda(\epsilon_x + \epsilon_z),\ \sigma_{yz} = \mu\epsilon_{yz} \\ \sigma_z &= (\lambda + 2\mu)\, \epsilon_z + \lambda(\epsilon_x + \epsilon_y),\ \sigma_{zx} = \mu\epsilon_{zx} \end{aligned}\right] \quad \ldots(2.3)$$

the stresses can be obtained in terms of displacements.

3. Equations of Equilibrium

Applying Hamilton's energy principle the equations of equilibrium can be derived to be

$$\left.\begin{aligned} &m_{x,x} - q_x = 0 \\ &r_{x,x} - 3p_x = 0 \\ &q_{x,x} + p = 0 \\ &4p_{x,x} - 8m_z + h^2 p = 0 \end{aligned}\right] \qquad \ldots(3.1)$$

where

$$(m_x, m_z, q_x, r_x, p_x) = \int_{-h/2}^{h/2} (z\sigma_x, z\sigma_z, \sigma_{zx}, z^3\sigma_x, z^2\sigma_{zx})\, dz \qquad \ldots(3.2)$$

and the end conditions are the vanishing of one member of each of the following pairs ;

$$(m_x, u_1), (r_x, u_3), (q_x, w_0), (p_x, w_2). \qquad \ldots(3.3)$$

With the help of eqns. (2.2), (2.3) and (3.1) the equations of equilibrium can be obtained in terms of displacements. Making the equations thus obtai. ...upled in displacement variables by matrix elimination method and then reducing to non-dimensional form, we get

$$\left.\begin{aligned} D\nabla^4 U_3 &= 16800\,(2 - \nu)\,(1 - 2\nu)\,(1 + \nu) \\ D^3\nabla^4 U_1 &= -\,100800\,(1 - 2\nu)\,(1 - \nu^2) \\ D^2\nabla^4 W_2 &= 50400\,\nu(1 - 2\nu)\,(1 + \nu)\,H \\ D^4\nabla^4 W_0 &= 100800\,(1 - 2\nu)\,(1 - \nu^2)\,H \end{aligned}\right] \qquad \ldots(3.4)$$

where

$$\left.\begin{aligned} &D \equiv H\frac{d}{dx}, \ \nabla^4 \equiv (1 - \nu^2)\,D^4 - 120(1 - \nu)\,D^2 + 8400\,(1 - 2\nu) \\ &X = x/a,\ H = h/a,\ U_3 = Eu_3/p,\ U_1 = Eu_1/p \\ &W_2 = Ew_2/ap,\ W_0 = Ew_0/ap. \end{aligned}\right] \qquad \ldots(3.5)$$

4. SOLUTION

The general solution of eqns. (3.4) can be obtained as

$$\left.\begin{aligned} U_3 &= [\alpha_1(e^{\xi}D_1 - e^{-\xi}D_2) + \alpha_2(e^{\eta}D_3 - e^{-\eta}D_4) + \alpha_3 H^3 D_8 + \alpha_4 X]/H \\ U_1 &= [\beta_1(e^{\xi}D_1 - e^{-\xi}D_2) + \beta_2(e^{\eta}D_3 - e^{-\eta}D_4) - HD_6 - 2HXD_7 \\ &\quad - H(3X^2 + H^2\beta_3)D_8 - \beta_4 X - 4\beta_5 X^3/H^2]/H \\ W_2 &= -\,\gamma_1(e^{\xi}D_1 + e^{-\xi}D_2) - \gamma_2(e^{\eta}D_3 + e^{-\eta}D_4) \\ &\quad + \gamma_3 H^2(D_7 + 3XD_8)\,\gamma_4 H + \gamma_5 X^2/H \\ W_0 &= e^{\xi}D_1 + e^{-\xi}D_2 + e^{\eta}D_3 + e^{-\eta}D_4 + D_5 \\ &\quad + XD_6 + X^2D_7 + X^3D_8 + X^4\beta_5/H^3 \end{aligned}\right\} \qquad \ldots(4.1)$$

where

$$
\left.\begin{aligned}
&\alpha_1 = A\alpha^3/N_1,\ \alpha_2 = A\beta^3/N_2,\ \alpha_3 = (2-\nu)/(1-\nu),\\
&\alpha_4 = 2(2-\nu)(1+\nu),\ A = 280(1-\nu)(1-2\nu),\\
&\beta_1 = (B_a\alpha^2 - B_b)\,\alpha/N_1,\ \beta_2 = (B_a\beta^2 - B_b)\,\beta/N_2,\ \beta_3 = 1.5/(1-\nu),\\
&\beta_4 = 3(1+\nu),\ \beta_5 = (1-\nu^2)/2,\ B_a = 6(1-\nu)(7-10\nu),\\
&B_b = 1680(1-2\nu)\,\nu,\ \gamma_1 = (C_a\alpha^2 - C_b)/N_1,\ \gamma_2 = (C_a\beta^2 - C_b)/N_2,\\
&\gamma_3 = \nu/(1-\nu),\ \gamma_4 = 3(3\nu^2 - 4\nu + 4)/20(1-\nu),\ \gamma_5 = 3\nu(1+\nu),\\
&C_a = 120(1-\nu)(5+14\nu),\ C_b = 100800(1-2\nu),\ N_1 = D_a\alpha^2 - D_b,\\
&N_2 = D_a\beta^2 - D_b,\ D_a = 6(1-\nu)(13+10\nu),\ D_b = 1680(1-2\nu)(5-\nu),\\
&\xi = \alpha X/H,\ \eta = \beta X/H,\ \text{where } \nu \text{ is the Poisson's ratio.}
\end{aligned}\right\}
\quad \ldots(4.2)
$$

$$
\begin{matrix}\alpha\\ \beta\end{matrix} = \sqrt{[20\{3 \pm \sqrt{(42\nu - 12)}\}]}/\sqrt{(1-\nu)}
$$

and D_i, $i = 1, 2, \ldots, 8$ are the arbitrary constants to be determined by the end conditions. It is clear from above that α and β are real if $\nu \geqslant 2/7$ and complex if $\nu < 2/7$.

The end conditions for a beam clamped at both the ends are given by

$$
U_3 = U_1 = W_2 = W_0 = 0 \text{ at } X = 0 \text{ and } 1. \quad \ldots(4.3)
$$

Substituting these end conditions in equations (4.1), we get eight linear simultaneous equations which can be solved for the arbitrary constants D_i.

The non-dimensional stresses, shear force and bending moment are given by the expressions :

$$
\left.\begin{aligned}
&T_x = \sigma_x/p = [(1-\nu)\,U_z + \nu W_z]/N\\
&T_y = \sigma_y/p = \nu(U_z + W_z)/N\\
&T_z = \sigma_z/p = [\nu U_z + (1-\nu)\,W_z]/N\\
&T_{zx} = \sigma_{zx}/p = [U_1 + W_{0,x} + Z^2(3U_3 + W_{2,x})]/2(1+\nu)\\
&Q_x = q_x/ap = H[12(U_1 + W_{0,x}) + 3U_3 + W_{2,x}]/24(1+\nu)\\
&M_z = m_x/a^2p = H[(1-\nu)\,H^2(20U_{1,x} + 3U_{3,x}) + 40\nu W_2]/240N\\
&\text{where } U_z = ZH(U_{1,x} + Z^2U_{3,x}),\ W_z = 2ZW_2/H,\\
&\qquad N = (1+\nu)(1-2\nu) \text{ and } Z = z/h
\end{aligned}\right\}
\quad \ldots(4.4)
$$

If we neglect u_3 and w_2 from eqns. (2.1) and proceed in a similar manner as above and also introduce the Timoshenko constant in the shear force, the solution based upon Timoshenko theory can be obtained. For obtaining the solution for classical theory u_3 and w_2 are to be neglected and u_1 is taken equal to $-w_{0,x}$ in eqn. (2.1).

5. Numerical Results and Discussions

The transverse displacement parameter W and the stress parameters T_x and T_{zx} are computed for various values of X, H and Z for the present theory (P-theory), Timoshenko theory (T-theory) and classical theory (C-theory) and a comparison between them is made. The value of Timoshenko constant for T-theory is taken equal to $10(1 + \nu)/(12 + 11\nu)$ as given by Cowper (1966). The value of ν is taken equal to 0.29.

W for the neutral surface versus X is plotted in Fig. 1 which shows that W in T-theory is larger than W in C-theory and smaller than W in P-theory. This difference is greatest at the middle point of the beam.

W for the neutral surface versus H is plotted in Fig. 2. It shows that as H increases W decreases, first rapidly and then slowly, in all the three theories. As shown in Table I the percentage differences in W of P- and T-theories and T and C-theories increases with increasing H but the difference in P- and T-theories does not increase so rapidly as it increases in T- and C-theories. Therefore in the beginning the difference in P- and T-theories is more than the difference in T- and C-theories, but later on the thing is reversed.

In T- and C-theories W remains constant with variation of Z but in P-theory it varies in accordance with eqn. (2.1). In Table II, W of the upper or lower facings and W for the neutral surface in P-theory are given for various values of X. It is found that near the ends of the beam W of facings is slightly greater than the W of the neutral surface whereas in the middle of the beam W of facings is slightly lesser than the W of neutral surface. It is due to the fact that during bending in clamped end conditions, the beam near ends stretches above the neutral surface and contracts below it, whereas in the middle, the beam contracts above the neutral surface and stretches below it.

In Fig. 3, the behaviour of normal stress T_x at the upper facing is shown with respect to X. As we move from the left end towards the middle of the beam the value of T_x changes from negative to positive and it is zero near $X = 0.2$. Values of T_x obtained in P-theory are always greater than that of T-theory. The difference in T_x of two theories increases as we move from ends towards the middle of the beam.

T_x versus H is plotted in Fig. 4. As H increases T_x decreases, first rapidly and then slowly. The difference in T_x of P- and T-theories increases with increasing H.

The variation of T_x with Z is shown in Fig. 5. T_x in T-theory varies linearly whereas in P-theory it varies cubically. The effect of cubic term is not so evident for small H. But for large H it is quite evident as it can be seen from the graphs plotted for $H = 0.1$ and 0.2.

In Fig. 6, T_{zx} versus Z is plotted for various values of X. In P-theory T_{zx} varies parabolically and it is zero on the upper and lower facings and maximum at the surface, whereas in T-theory T_{zx} remains constant throughout the thickness.

TABLE I

Variation of percentage differences in W of P- and T-theories and T- and C-theories with respect to H

H	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
Percentage difference in W between P- and T-theories	17.03	17.47	18.04	18.68	19.36	20.07	20.77	21.45
Percentage difference in W between T- and C-theories	5.42	9.25	13.74	18.65	23.79	28.96	34.03	38.91

TABLE II

Variation of W of the upper or lower facings and W of the neutral surface for P-theory with respect to X

X	0.1	0.2	0.3	0.4	0.5
W at upper or lower facing	20.4976	59.7256	100.4664	129.9992	140.6916
W at neutral surface	20.1392	59.6946	100.6693	130.3424	141.0815

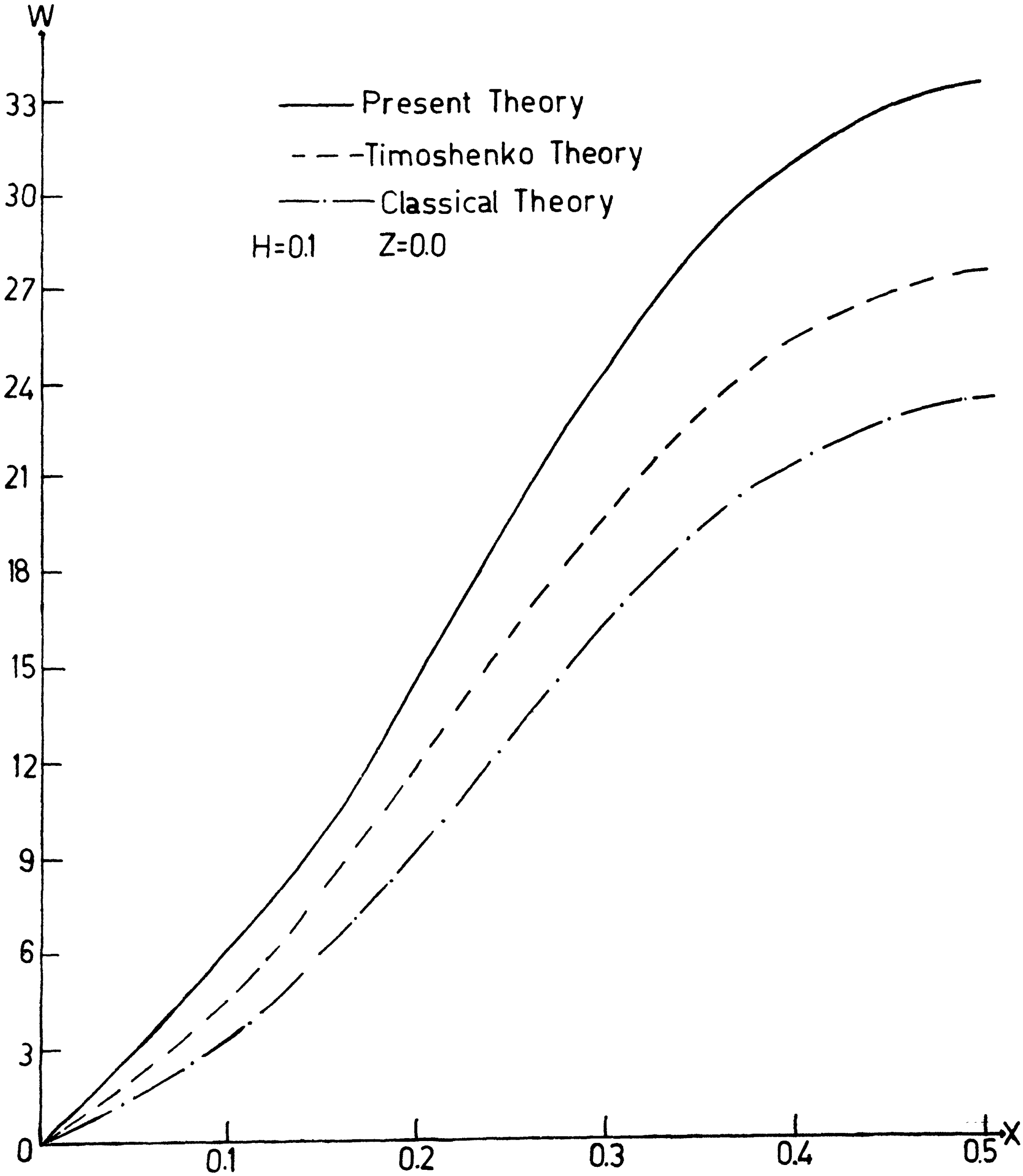

FIG. 1. Plot of W versus X.

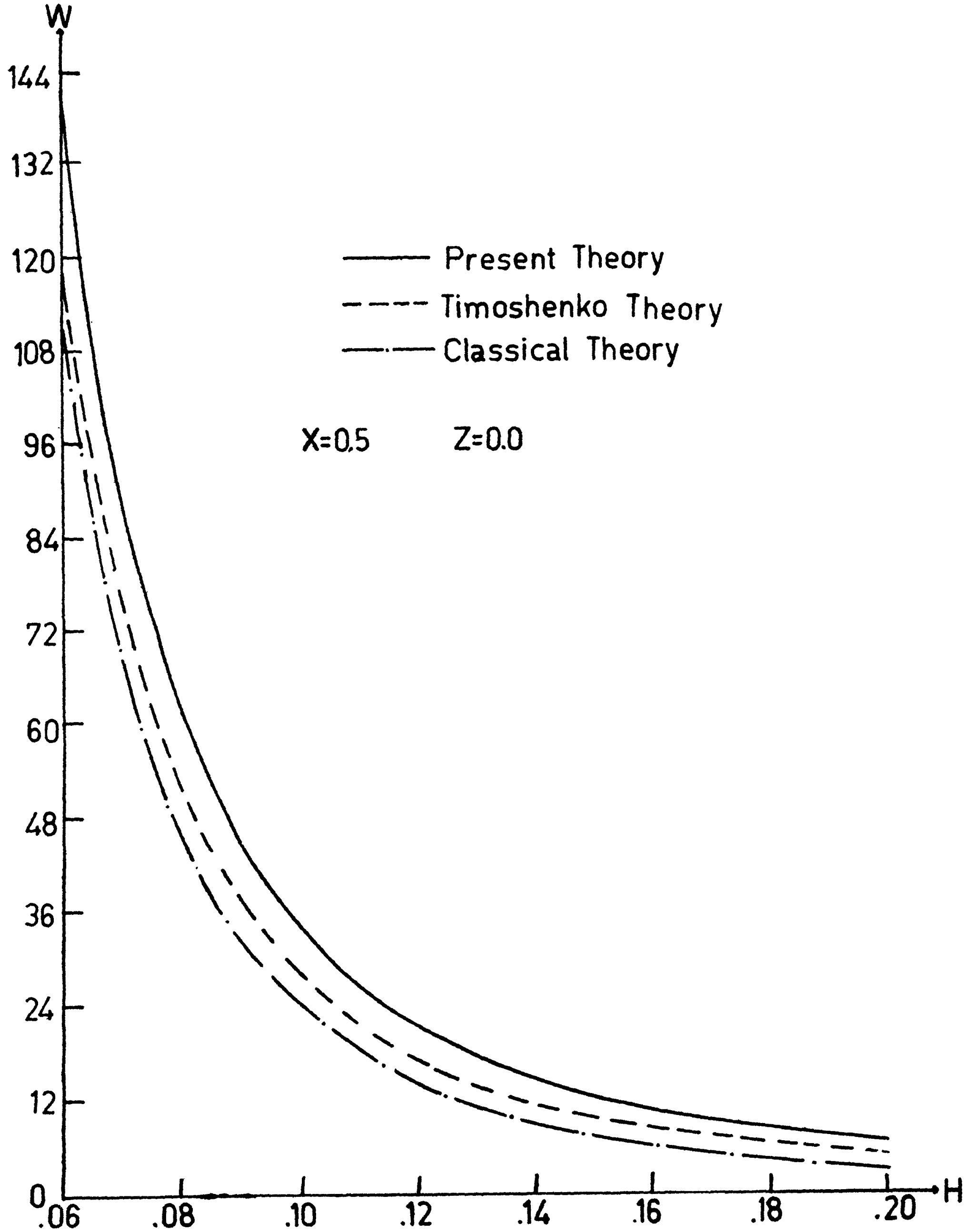

FIG. 2. Plot of W versus H.

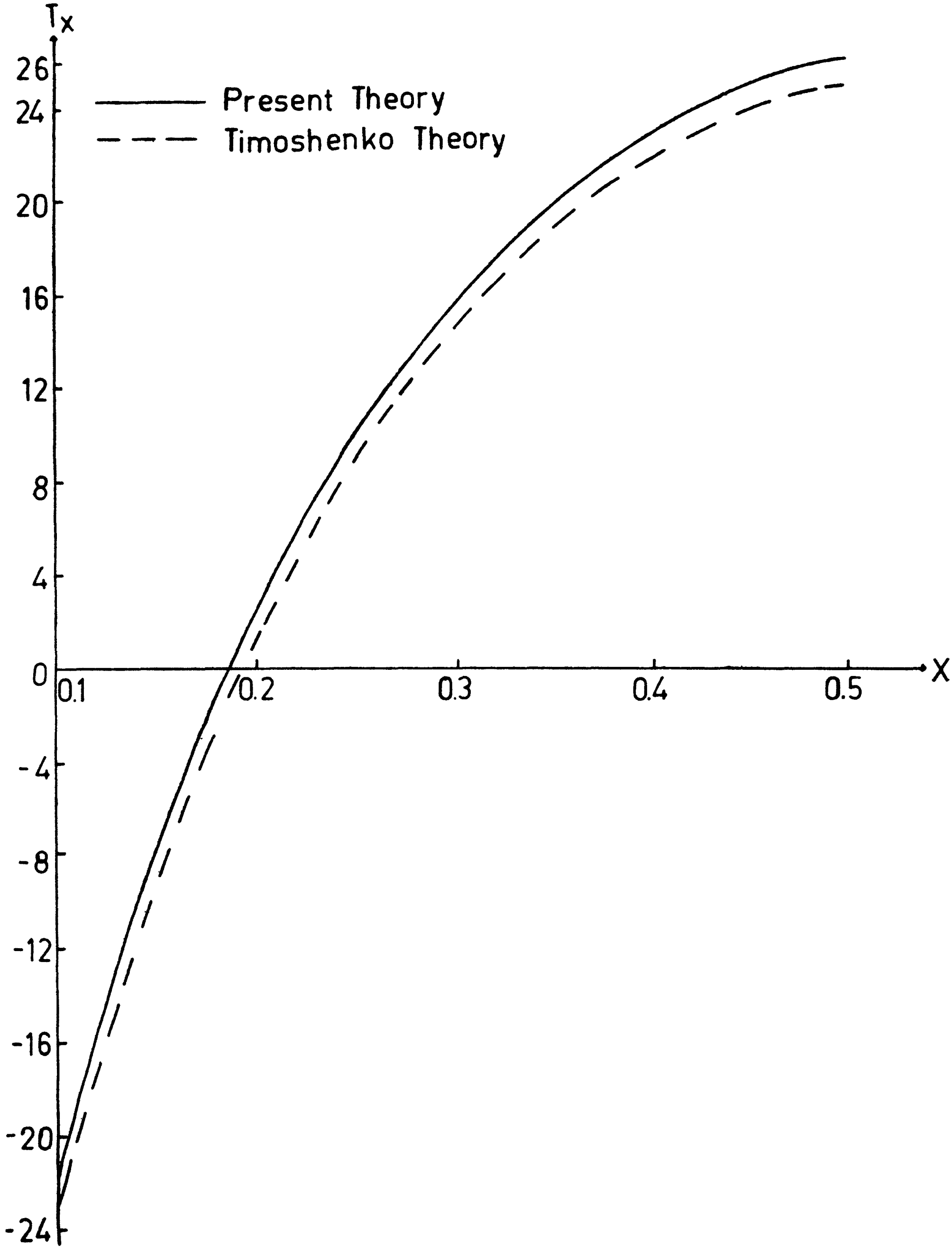

FIG. 3. Plot of T_x versus X.

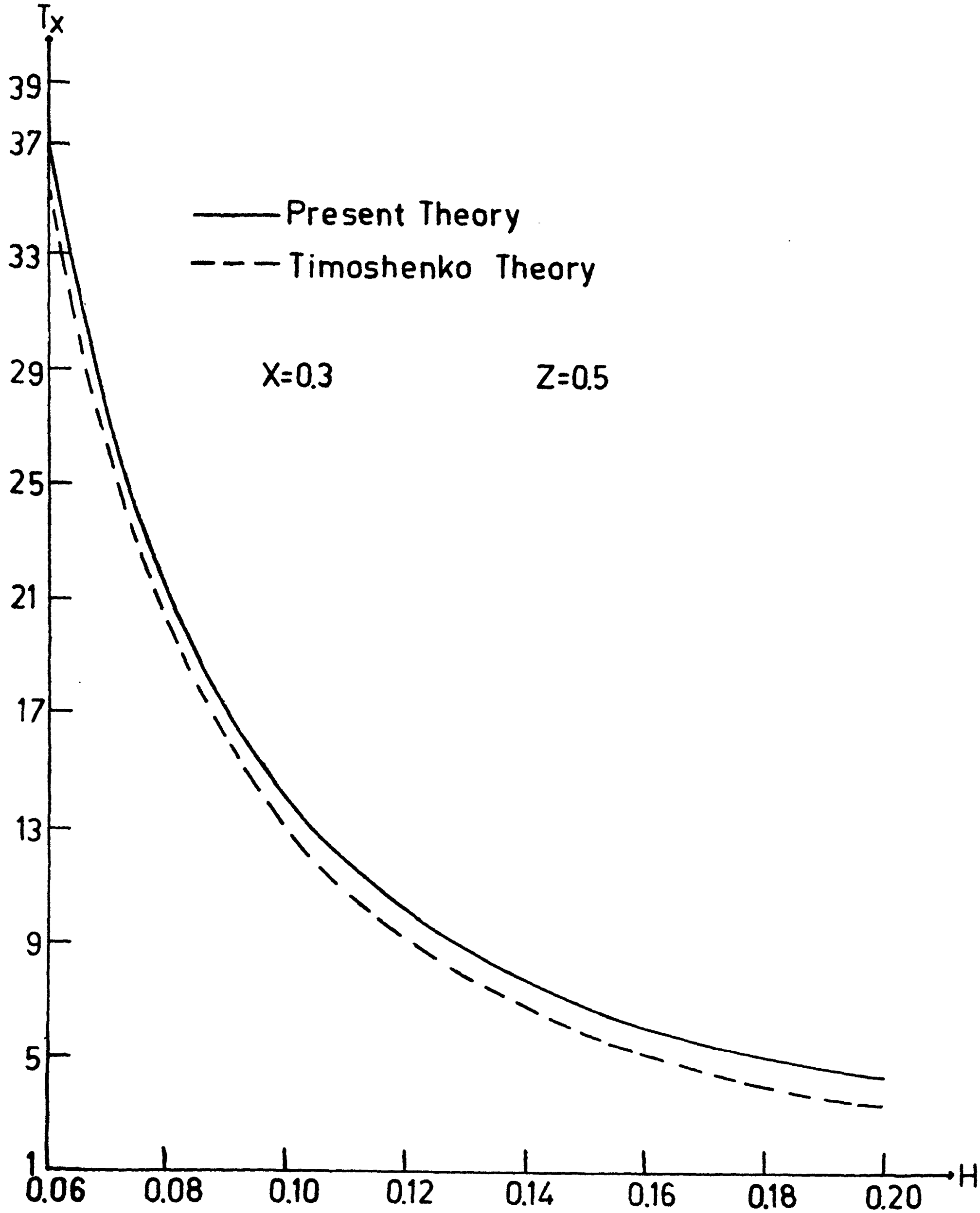

FIG. 4. Plot of T_x versus H.

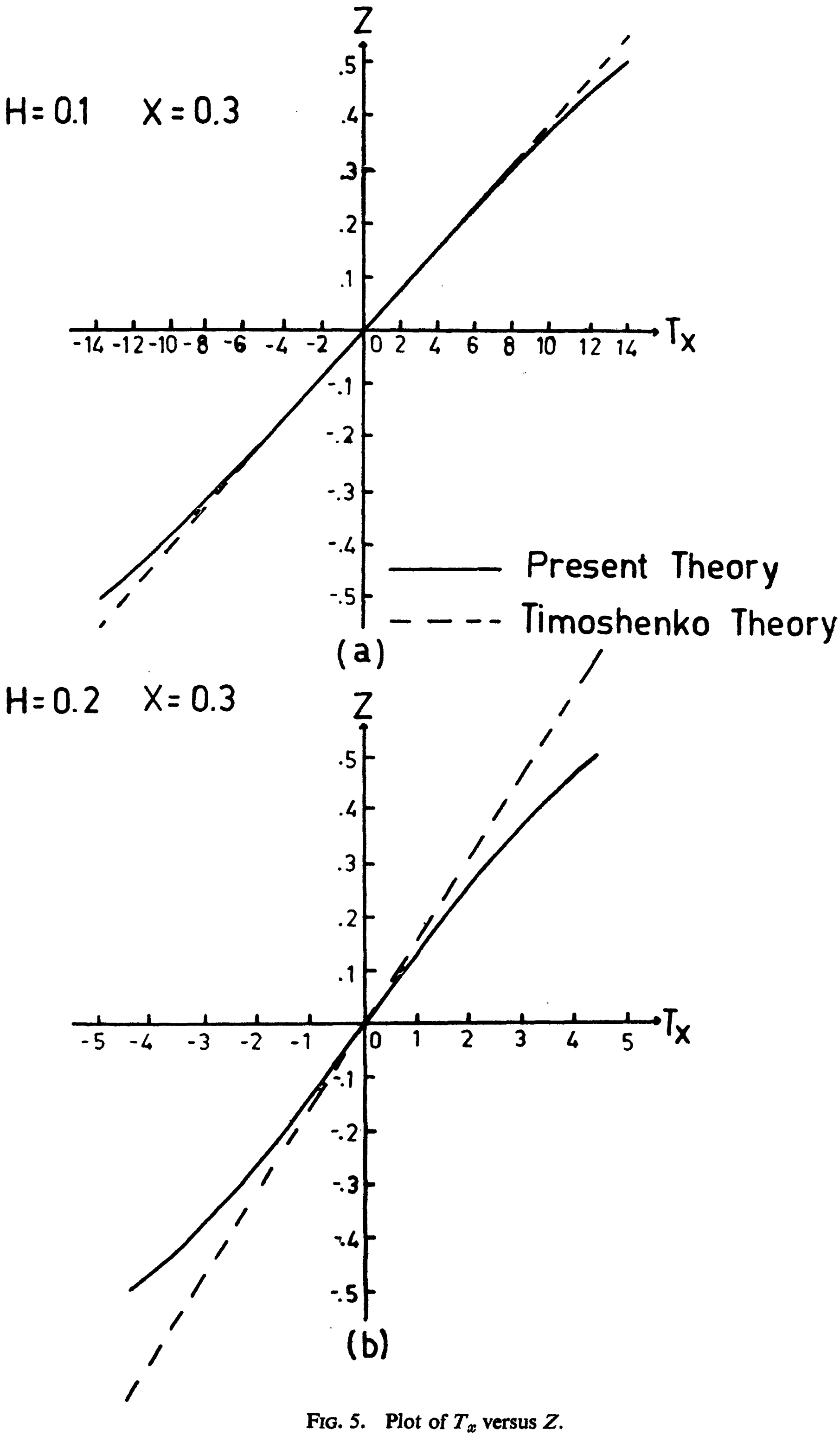

FIG. 5. Plot of T_x versus Z.

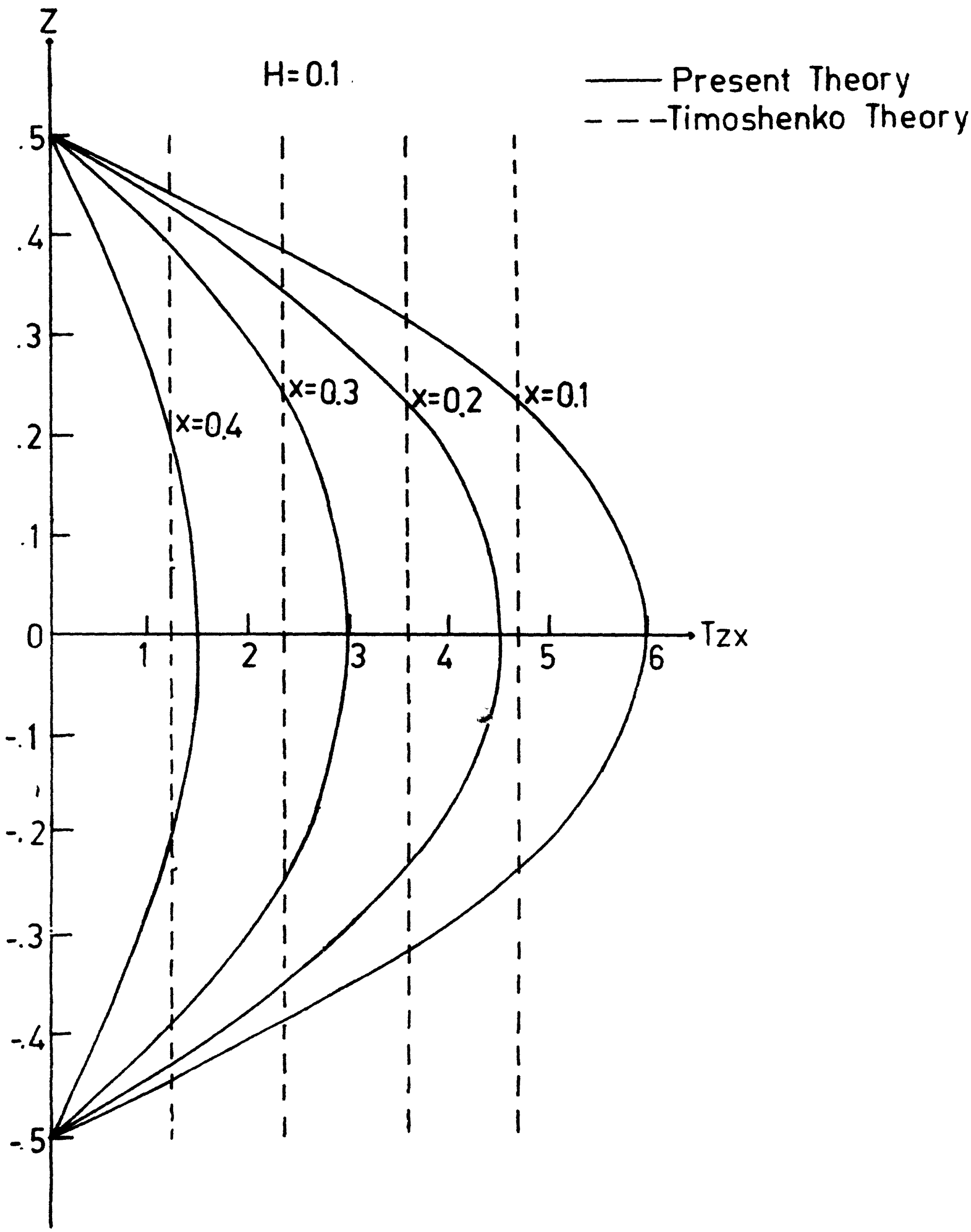

FIG. 6. Plot of T_{zx} versus Z.

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