ON SOME TRANSFORMATIONS OF TRIPLE HYPERGEOMETRIC SERIES $F^{(3)}$

by M. A. PATHAN, *Department of Mathematics, Al Fateh University, P.O. Box* 656, *Tripoli, Libya*

(*Received* 22 *July* 1977; *after revision* 26 *October* 1977)

A general triple hypergeometric series $F^{(3)}$, which is a generalisation of Lauricella's fourteen hypergeometric functions $F_1, \ldots, F_{14}$ and Srivastava's three additional functions H_A, H_B and H_C, has been given by Srivastava. In this paper, we obtain some transformations of $F^{(3)}$ and deduce some reduction formulas and expansions of hypergeometric functions of one, two and three variables as special cases.

§1. A unification of Lauricella's (1893, p. 114) fourteen hypergeometric functions $F_1, \ldots, F_{14}$ of three variables and Srivastava's (1967b) three additional functions H_A, H_B, H_C was introduced by Srivastava (1967a, p. 428) in the form of a triple hypergeometric series $F^{(3)}$ defined as

$$F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\; x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}\,((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}\,((c))_m\,((c'))_n\,((c''))_p\, x^m y^n z^p}{((e))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}\,((h))_m\,((h'))_n\,((h''))_p\, m!\, n!\, p!} \quad \ldots(1.1)$$

where (a) is a sequence of A parameters $a_1, \ldots, a_A$, and

$$((a))_n = \prod_{j=1}^{A} \frac{\Gamma(a_j + n)}{\Gamma(a_j)}, \text{ and similarly for } (b), ((b))_n, \text{ etc.}$$

In this paper, we obtain some transformation formulas for $F^{(3)}$ in the forms :

$$F^{(3)}\left[\begin{matrix}a::\; b;-;-:g-f;d;c-b;\\ c::-;-;-:\;g\;;e;\;-\;;\end{matrix}\; \frac{x}{z+1}, \frac{y}{z+1}, \frac{z}{z+1}\right] =$$

(*equation continued on p.* 372)

$$= (z+1)^{c-b} F^{(3)}\left[\begin{matrix} -::a,b;-;-:g-f;d;c-a,c-b; \\ c::-;-;-:\ g\ ;e;-; \end{matrix} \quad \frac{x}{z+1}, \frac{y}{z+1}, -z\right] \quad ...(1.2)$$

$$=(z+1)^{a}\ F^{(3)}\left[\begin{matrix} a,b::-;-;-:d;-;f; \\ c\ ::-;-;-:e;-;g; \end{matrix} \quad y, x-z, -x\right] \quad ...(1.3)$$

with the help of an integral evaluated in §2. We then deduce some known and unknown expansions and reduction formulas for hypergeometric functions ${}_2F_1$, ${}_3F_2$, Appell's functions F_1, F_2, F_3 and hypergeometric function of three variables F_A. (For definitions, see Erdélyi 1953).

§2. The integral to be established is

$$I = \int_0^\infty t^{\lambda}\, e^{-(2z+(p-x-y))t/2}\, W_{K,\mu}(pt)\, M_{\sigma,\nu}(xt)\, M_{\eta,\xi}(yt)\, dt$$

$$= \frac{\Gamma(a \pm \mu)\, x^{\nu+(1/2)}\, y^{\xi+(1/2)}\, p^{\mu+(1/2)}}{\Gamma(a-K+\frac{1}{2})\,(z+p)^{a+\mu}}$$

$$\times F^{(3)}\left[\begin{matrix} a+\mu::a-\mu;-;-:\frac{1}{2}-\sigma+\nu;\frac{1}{2}-\eta+\xi;\mu-K+\frac{1}{2}; \\ a-K+\frac{1}{2}::\ -\ ;-;-:\ 2\nu+1;\ 2\xi+1\ ;-; \end{matrix} \quad \frac{x}{z+p}, \frac{y}{z+p}, \frac{z}{z+p}\right] \quad ...(2.1)$$

where $a = \lambda + \nu + \xi + \frac{5}{2}$, $\mathrm{Re}(a+\mu) > 0$ and $\mathrm{Re}(2z+p-x-y \pm p \pm x \pm y) > 0$.

Expressing Whittaker's functions $M_{\sigma,\nu}(xt)$ and $M_{\eta,\xi}(yt)$ in terms of ${}_1F_1$, expanding ${}_1F_1$ in series and integrating term by term with the help of the result (Erdélyi 1954a, p. 216(16)), we get

$$I = p^{\mu+(1/2)}\, x^{\nu+(1/2)}\, y^{\xi+(1/2)}$$

$$\times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\Gamma(a \pm \mu + r + s)\,(\frac{1}{2}-\sigma+\nu)_r\,(\frac{1}{2}-\eta+\xi)_s}{r!\,s!\,(2\nu+1)_r\,(2\xi+1)_s\,\Gamma(a-K+\frac{1}{2}+r+s)}\ \frac{x^r y^s}{(z+p)^{a+r+s+\mu}}$$

$$\times {}_2F_1\left(\begin{matrix} a+r+s+\mu,\ \mu-K+\frac{1}{2} \\ a-K+\frac{1}{2}+r+s \end{matrix} ;\ \frac{z}{z+p}\right). \quad ...(2.2)$$

Again expressing ${}_2F_1$ in series and interpreting by (1.1), we arrive at (2.1).

If we apply a relation (Erdélyi 1953, p. 105)

$$ {}_2F_1\left(\begin{matrix}\alpha, \beta \\ \\ \gamma\end{matrix} ; z\right) = (1-z)^{-\beta}\, {}_2F_1\left(\begin{matrix}\gamma-\alpha, \beta \\ \\ \gamma\end{matrix} ; \frac{z}{z-1}\right) $$

to (2.2) and interpret with the help of (1.1), then we get

$$ I = \frac{\Gamma(a \pm \mu)\, x^{\nu+(1/2)}\, y^{\xi+(1/2)}\, p^{K}}{\Gamma(a-K+\frac{1}{2})\,(z+p)^{a+K-(1/2)}} $$

$$ \times F^{(3)}\left[\begin{matrix} - :: a\pm\mu; -; - : \frac{1}{2}-\sigma+\nu; \frac{1}{2}-\eta+\xi; \frac{1}{2}-K\pm\mu; \\ \qquad\qquad\qquad\qquad \frac{x}{z+p}, \frac{y}{z+p}, -\frac{z}{p} \\ a-K+\frac{1}{2} :: - ; -; - : 2\nu+1; 2\xi+1; -; \end{matrix}\right]. \qquad ...(2.3) $$

Equating (2.3) with (2.1) and adjusting the parameters, we get a transformation formula (1.2).

On the other hand, if we express $M_{\sigma,\nu}(xt)$ in the integral I in terms of ${}_1F_1$, expand ${}_1F_1$ and $e^{-(z-x)t}$ in series and integrate term by term with the help of the result (Erdélyi 1954b, p. 410(43)), we obtain

$$ I = \frac{\Gamma(a \pm \mu)\, x^{\mu+(1/2)}\, y^{\xi+(1/2)}}{\Gamma(a-K+\frac{1}{2})\, p^{a-(1/2)}} $$

$$ \times F^{(3)}\left[\begin{matrix} a\pm\mu :: -; -; - : \frac{1}{2}+\xi-\eta; -; \frac{1}{2}+\sigma+\nu; \\ \qquad\qquad\qquad\qquad \frac{y}{p}, \frac{x-z}{p}, -\frac{x}{p} \\ a-K+\frac{1}{2} :: -; -; - : 2\xi+1; -; 2\nu+1 ; \end{matrix}\right]. \qquad ...(2.4) $$

Again, equating (2.1) and (2.4) and adjusting the parameters, we get (1.3).

§3. In the special case when $x \to 0$ and $b = e$, (1.2) reduces to a known result (Appell and Kampé de Fériet 1926, p. 36)

$$ F_1\left(a, d, c-b, c ; \frac{y}{z+1}, \frac{z}{z+1}\right) $$
$$ = (z+1)^{c-b} F_3\left(a, c-a, d, c-b, c ; \frac{y}{z+1}, -z\right) \qquad ...(3.1) $$

where F_1 and F_3 are Appell's hypergeometric functions.

For $z = x$, (1.3) reduces to

$$F^{(2)}\begin{bmatrix} a, b : d ; f ; & \\ & y, -x \\ c \quad : e ; g ; & \end{bmatrix} = (x+1)^{c-a-b}$$

$$\times F^{(3)}\begin{bmatrix} - : : a, b ; - ; - : g-f ; d ; c-a, c-b ; & \\ & \dfrac{x}{1+x}, \dfrac{y}{1+x}, -x \\ c : : - ; - ; - : g ; e ; - ; & \end{bmatrix} \quad \ldots(3.2)$$

where $F^{(2)}$ is Kampé de Fériet's hypergeometric function of two variables (Appell and Kampé de Fériet 1926) in the contracted notation of Burchnall and Chaundy (1941, p. 112).

(3.2) is equivalent to the expansion formula

$$\sum_{r=0}^{\infty} \frac{(c-a)_r (c-b)_r}{r!\,(c)_r} (-x)^r F^{(2)}\begin{bmatrix} a, b : g-f ; d ; & \\ & \dfrac{x}{1+x}, \dfrac{y}{1+x} \\ c+r : g ; e ; & \end{bmatrix}$$

$$= (1+x)^{-c+a+b} F^{(2)}\begin{bmatrix} a, b : d ; f ; & \\ & y, -x \\ c : e ; g ; & \end{bmatrix}. \quad \ldots(3.3)$$

For $c = a$, (3.3) reduces to a known transformation of Appell's function F_2 in the same series of F_2 (Erdélyi 1953, p. 240) in the form

$$F_2(b, d, f, e, g ; y, -x) = (1+x)^{-b} F_2\left(b, g-f, d, g, e; \frac{x}{1+1}, \frac{y}{1+x}\right). \quad \ldots(3.4)$$

On taking $y \to 0$, (3.3) gives us

$$\sum_{r=0}^{\infty} \frac{(c-a)_r (c-b)_r}{r!\,(c)_r} (-x)^r {}_3F_2\left(\begin{matrix} a, b, g-f \\ c+r, g \end{matrix} ; \frac{x}{1+x}\right)$$

$$= (x+1)^{-c+a+b} {}_3F_2\left(\begin{matrix} a, b, f \\ c, g \end{matrix} ; -x\right). \quad \ldots(3.5)$$

On using a transformation (Pathan 1977)

$$\frac{\Gamma(a \pm \nu)}{\Gamma(a+c)} F^{(2)}\begin{bmatrix} a \pm \nu : d ; f ; & \\ & \dfrac{z}{y}, -\dfrac{x}{y} \\ a+c : e ; g ; & \end{bmatrix} =$$

(*equation continued on p.* 375)

$$= \sum_{\nu,\,-\nu} \frac{\Gamma(-2\nu)\,\Gamma(a+\nu)}{\Gamma(c-\nu)}\, y^{a+\nu}$$

$$\times F_A(a+\nu\,; g-f, c+\nu, d\,; g, 2\nu+1, e\,; x, y, z) \qquad ...(3.6)$$

where F_A is hypergeometric function of three variables (see Erdélyi 1954b, p. 445)), (3.2) yields a transformation of $F^{(3)}$ into F_A in the form

$$F^{(3)}\left[\begin{matrix} - :: a\pm\nu\,; -; - : g-f\,; d\,; c\pm\nu\,; \\ \\ a+c :: - \;; -; - : g\;; e\,; -\,; \end{matrix}\;\frac{x}{y+x}, \frac{z}{y+x}, \frac{-x}{y}\right]$$

$$= y^{c}(x+y)^{a-c}\frac{\Gamma(a+c)}{\Gamma(a\pm\nu)}\sum_{\nu,\,-\nu} y^{\nu}\,\frac{\Gamma(-2\nu)\,\Gamma(a+\nu)}{\Gamma(c-\nu)}$$

$$\times F_A(a+\nu\,; g-f, c+\nu, d\,; g, 2\nu+1, e\,; x, y, z). \qquad ...(3.7)$$

For $z \to 0$ and $f = 0$, (3.7) reduces to

$$(x+y)^{c-a}\,F_3\!\left(a+\nu, c+\nu, a-\nu, c-\nu, a+c\,;\ \frac{x}{x+y}, \frac{-x}{y}\right)$$

$$= \frac{\Gamma(a+c)}{\Gamma(a\pm\nu)}\sum_{\nu,\,-\nu}\frac{\Gamma(-2\nu)\,\Gamma(a+\nu)}{\Gamma(c-\nu)}\,y^{c+\nu}$$

$$\times F_2(a+\nu, g, c+\nu, g, 2\nu+1\,; x, y)$$

$$= \frac{\Gamma(a+c)}{\Gamma(a\pm\nu)}\sum_{\nu,\,-\nu}\frac{\Gamma(-2\nu)\,\Gamma(a+\nu)\,y^{c+\nu}}{\Gamma(c-\nu)\,(1-x)^{a+\nu}}$$

$$\times\ {}_2F_1\left(\begin{matrix} a+\nu, c+\nu \\ \\ 2\nu+1 \end{matrix}\ ;\ \frac{y}{1-x}\right) \qquad ...(3.8)$$

by using a reduction formula (Erdélyi 1953, p. 238(2)).

For $z \to 0$, (1.3) yields a reduction formula

$$F^{(3)}\left[\begin{matrix} a\pm\nu :: - ; - ; - : d\,; - ; f\,; \\ \\ a+c :: - ; - ; - : e\,; - ; g\,; \end{matrix}\ y, x, -x\right]$$

$$= F^{(2)}\left[\begin{matrix} a\pm\nu : g-f\,; d\,; \\ \\ a+c : g\;; e\,; \end{matrix}\ x, y\right] \qquad ...(3.9)$$

which can be written in an alternate form :

$$\sum_{r=0}^{\infty} \frac{(a \pm v)_r}{r!\,(a+c)_r}\, x^r F^{(2)}\left[\begin{matrix} a \pm v + r : d\,; f\,; \\ a + c + r : e\,; g\,; \end{matrix}\; y, -x\right]$$

$$= F^{(2)}\left[\begin{matrix} a \pm v : g - f\,; d\,; \\ a + c : g\,;\ e\,; \end{matrix}\; x, y\right]. \qquad \ldots(3.10)$$

For $d = e$, (3.9) gives an expansion

$$\sum_{n=0}^{\infty} \frac{(a \pm v)_n\,(f)_n}{n!\,(g)_n}\,(-x)^n\, {}_2F_1\left(\begin{matrix} a \pm v + n \\ a + c + n \end{matrix}\; ; y + x\right)$$

$$= F^{(2)}\left[\begin{matrix} a \pm v : g - f\,; -\,; \\ a + c :\ g\ ; -\,; \end{matrix}\; x,\ y\right] \qquad \ldots(3.11)$$

by using a well-known formula (Appell and Kampé de Fériet 1926, pp. 13, 14)

$$F^{(2)}\left(\begin{matrix} a, b : -\,; -\,; \\ c\ : -\,; -\,; \end{matrix}\; x, y\right) = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\; ; x + y\right]. \qquad \ldots(3.12)$$

ACKNOWLEDGEMENT

The author is thankful to the referee for his valuable suggestions.

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