THE MIXED SEMIGROUP RELATION

by HARSHINDER SINGH, *Department of Statistics, Panjab University, Chandigarh* 160014

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Let X denote a Banach space, and let $B(X)$ denote the family of bounded linear operators on X. Let $R^+ = [0, \infty)$. Let $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$, be a one parameter family of operators. The family $\{S(t)\}$ is called a 'mixed semigroup' if it satisfies the equation : $S(s + t) - S(s)\, S(t) = \alpha(S(s) - T(s))\,.\,(S(t) - T(t))$, where α is any real number and $\{T(t); t \in R^+\}$, $T : R^+ \to B(X)$, is a known semigroup of operators. Let A_0 be the infinitesimal generator of $\{T(t)\}$. Let $A = \lim\limits_{h \to 0} (S(h) - I)/h$, be the first infinitesimal generator of $\{S(t)\}$. For the case $\alpha \neq -1$, the representation $S(t) = (\alpha T(t) + T_1(t))/(1 + \alpha)$ holds in the strong operator topology, where $\{T_1(t)\}$ is the semigroup generated by $(1 + \alpha) A - \alpha A_0$. For the case $\alpha = -1$, the representation $S(t) = T(t) + t(A - A_0)\, T(t)$, holds in the uniform operator topology. For the case $\alpha \neq -1$, the generation theorem can also be established under the usual boundedness conditions for the associated resolvent operator.

1. INTRODUCTION

Let X be a Banach space, and let $R^+ = [0, \infty)$. Let $B(X)$ denote the family of bounded linear operators on X. The family $\{T(t); t \in R^+\}$ $T: R^+ \to B(X)$, is called a 'semigroup of operators' if

$$T(s + t) = T(s)\, T(t),\ s, t \in R^+ \qquad \ldots(1)$$

with $T(0) = I$, the identity operator. The semigroup of operators has been extensively studied by Hille, Phillips and Yosida and is presented in the treatise of Hille and Phillips (1957). Buche and Vasudeva (1976) introduced the generalized Cauchy equation. The family $\{S(t); t \in R^+\}$, $S: R^+ \to B(X)$, is said to satisfy a generalized Cauchy equation if

$$S(t + s) = H(S(s), S(t)),\ s, t \in R^+ \qquad \ldots(2)$$

where $H: B(X) \times B(X) \to B(X)$, is a function. Note that the semigroup is a particular case of this $\{S(t)\}$ when

$$H(S(s), S(t)) = S(s)\, S(t).$$

In this paper we investigate the properties of the solution of an operator valued functional equation which is an interesting particular case of the generalized

Cauchy relation and we shall call this operator family $\{S(t)\}$ as the mixed semigroup, and is defined by the equation

$$S(s + t) - S(s) S(t) = \alpha(S(s) - T(s)) (S(t) - T(t)) \qquad ...(3)$$

$S(0) = I$, where α is any real number and $\{T(t), t \in R^+\}$, $T : R^+ \to B(X)$, is a known regular semigroup of operators. For $\alpha = 0$, $\{S(t)\}$ reduces to the usual semigroup.

We shall discuss eqn. (3) for two different cases $\alpha \neq -1$ and $\alpha = -1$. For the case $\alpha \neq -1$, the solution of eqn. (3) in the strong operator topology is analogous to the case mentioned by Feller (1968), who seems to be the first to use the name mixed semigroup. This case is discussed in the strong operator topology in Section 2. In Section 3, we have investigated the problem for the case $\alpha = -1$ in the uniform operator topology.

2. Representation and Generation of Mixed Semigroup for $\alpha \neq -1$

Let $\{S(t) : t \in R^+\}$, $S : R^+ \to B(X)$, be a continuous family of operators satisfying the equation (3) with $\alpha \neq -1$. Let A_0 be the infinitesimal generator of the known semigroup $\{T(t)\}$ of operators. Let

$$A_h = \frac{(S(h) - I)}{h}, h > 0. \qquad ...(4)$$

The first infinitesimal generator A of $\{S(t)\}$ is defined by

$$Af = \lim_{h \to 0} A_h f, \; f \in D(A) \qquad ...(5)$$

where $D(A) \subset X$, $D(A)$ is the set of elements $f \in X$, for which the above limit exists.

Let us denote by A_1 the operator $(1 + \alpha) A - \alpha A_0$. We have the following Lemma.

Lemma 1 — Let $\{S(t); t \in R^+\}$, $S : R^+ \to B(X)$ be a continuous family of operators satisfying eqn. (3) with $\alpha \neq -1$. Let

$$T_1(t) f = (1 + \alpha) S(t) f - \alpha T(t) f, \; f \in X \qquad ...(6)$$

then $\{T_1(t)\}$ is a continuous semigroup with the infinitesimal generator A_1.

Proof : $T_1(s + t) f = (1 + \alpha) S(s + t) f - \alpha T(s + t) f$, $t, s \in R^+$. Using (3) (with $\alpha \neq -1$), we get

$$T_1(s + t) f = (1 + \alpha) \Big(S(s) S(t) + \alpha(S(s) - T(s)) (S(t) - T(t)) \Big) f$$
$$- \alpha T(s) T(t) f$$

$$= (1 + \alpha) S(s) S(t) f + \alpha(1 + \alpha) S(s) S(t) f$$
$$- \alpha(1 + \alpha) S(s) T(t) f - \alpha(1 + \alpha) T(s) S(t) f$$
$$+ \alpha(1 + \alpha) T(s) T(t) f - \alpha T(s) T(t) f$$
$$= (1 + \alpha)^2 S(s) S(t) f - \alpha(1 + \alpha) S(s) T(t) f$$
$$- \alpha(1 + \alpha) T(s) S(t) f + \alpha^2 T(s) T(t) f$$
$$= ((1 + \alpha) S(s) - \alpha T(s)) ((1 + \alpha) (S(t) - \alpha T(t)) f$$
$$= T_1(s) T_1(t) f.$$

Moreover $T_1(0) f = (1 + \alpha) f - \alpha f = f$. This establishes the semigroup property of $\{T_1(t)\}$. $\{T_1(t)\}$ is continuous because $\{S(t)\}$ and $\{T(t)\}$ are continuous. The infinitesimal generator of $\{T_1(t)\}$ is given by

$$\lim_{h \to 0} (T_1(h) f - f)/h$$
$$= \lim_{h \to 0} ((1 + \alpha) S(h) f - \alpha T(h) f - f)/h$$
$$= \lim_{h \to 0} ((1 + \alpha) (S(h) f - f) - \alpha(T(h) f - f))/h$$
$$= A_1 f, \; f \in D(A_1).$$

This proves the Lemma.

The following theorem follows immediately from the Lemma 1.

Theorem 1 — Let $\{S(t); \; t \in R^+\}$, $S : R^+ \to B(X)$ be a continuous family of operators satisfying the eqn. (3). Then for $\alpha \neq -1$, the solution of (3) in the strong operator topology is of the form

$$S(t) f = (\alpha/(1 + \alpha)) T(t) f + (1/(1 + \alpha)) T_1(t) f. \quad \text{...(7)}$$

Remark 1 : For $\alpha = 1$, (7) reduces to the Feller's case (1968).

We now consider the resolvent operators

$$R(\lambda, A_0) = \int_0^\infty \exp(-\lambda t) T(t) dt$$

$$R(\lambda, A_1) = \int_0^\infty \exp(-\lambda t) T_1(t) dt.$$

From the semigroup theory, the above exist for real λ sufficiently large. We can characterize the mixed semigroup of operators (with $\alpha \neq -1$) by the following generation theorem.

Theorem 2 — A necessary and sufficient condition for a linear operator A on X to be the first infinitesimal generator of one-parameter family $\{S(t)\}$ of operators satisfying (3) with $\alpha \neq -1$, is that $A_1 = (1 + \alpha) A - \alpha A_0$ is a closed operator with domain dense in X, and there exists real numbers $M > 0$ and β such that for every real $\lambda > \beta$, λ in the resolvent set of A_1,

$$\| R(\lambda, A_1) \| \leqslant M(\lambda - \beta)^{-1}. \qquad ...(8)$$

PROOF : The necessity of the condition is immediate. In fact by the Lemma 1, $\{T_1(t)\}$ is a semigroup with generator A_1. From the semigroup theory, it follows that A_1 is closed operator with domain dense in X and there exist constants $M > 0$ and β such that $\| T_1(t) \| \leqslant M \exp(t\beta)$ and $\| R(\lambda, A_1) \| \leqslant M(\lambda - \beta)^{-1}$, $\lambda > \beta$.

As far as sufficiency is concerned we note that the condition (8) guarantees the existence of a semigroup of operator $\{T_1(t)\}$ with the infinitesimal generator A_1. Then $\{S(t)\}$ defined by

$$S(t) f = (\alpha/(1 + \alpha))\, T(t) f + (1/(1 + \alpha))\, T_1(t) f,\ \alpha \neq -1 \qquad ...(9)$$

satisfies the eqn. (3), with $\alpha \neq -1$. It is easy to see that A is the infinitesimal generator of $\{S(t)\}$.

Remark : The case $\alpha = 1$ also corresponds to a particular case of the exponential-cosine operator which is studied in the strong operator topology by Buche (1971).

3. REPRESENTATION AND GENERATION OF MIXED SEMIGROUP FOR THE CASE $\alpha = -1$

For $\alpha = -1$, the eqn. (3) reduces to

$$S(s + t) - S(s) S(t) = -(S(s) - T(s)(S(t) - T(t)) \qquad ...(10)$$

with $S(0) = I$. With the same notations as of Section 2, let A_0 be the infinitesimal generator of the known semigroup of operator $\{T(t)\}$, and A be the first infinitesimal generator of $\{S(t)\}$ satisfying (10). The following Lemma immediately follows from (10).

Lemma 2 — Let $\{S(t)\}$ be a continuous family of operators satisfying the eqn. (10). Then

$$dS(t)/dt = A_0 S(t) + (A - A_0) T(t) \qquad ...(11)$$

holds in the uniform operator topology. Moreover $dS(t)/dt$ is continuous in the uniform operator topology.

PROOF : $(dS(t)/dt)$

$$= \lim_{h \to 0} (S(t + h) - S(t))/h$$

$$= \lim_{h \to 0} (S(h) S(t) - (S(h) - T(h)(S(t) - T(t)) - S(t))/h$$

$$= \lim_{h\to 0} (S(h) - I)/h\, S(t) - \lim_{h\to 0} (S(h) - T(h))/h\, (S(t) - T(t))$$

$$= A\, S(t) - (A - A_0)(S(t) - T(t))$$

$$= A_0 S(t) + (A - A_0)\, T(t).$$

Continuity of $dS(t)/dt$ in the uniform operator topology follows from the continuity of $\{S(t)\}$ and $\{T(t)\}$.

We now have the following representation.

Theorem 3 — Let $\{S(t)\}$ be a continuous family of operators satisfying (10). Then the solution of (10), in the uniform operator topology, is of the form

$$S(t) = T(t) + t(A - A_0)\, T(t). \qquad \text{...(12)}$$

PROOF : $S(t) - T(t) = \int_0^t (d(T(t - z)\, S(z))/dz)\, dz$

$$= -\int_0^t A_0 T(t - z)\, S(z)\, dz$$

$$+ \int_0^t T(t - z)\, (A_0 S(z) + (A - A_0)\, T(z))\, dz$$

$$= \int_0^t (A - A_0)\, T(t)\, dz$$

$$= t(A - A_0)\, T(t).$$

Hence

$$S(t) = T(t) + t(A - A_0)\, T(t).$$

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