DEGREE OF APPROXIMATION BY CESÀRO MEANS OF FOURIER-LAGUERRE SERIES

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In this note a theorem is established on the degree of approximation to a function by Cesàro means of its Fourier-Laguerre series. The result is an extension of a classical theorem of Flett (1956) on Fourier-trigonometric series.

1. DEFINITIONS AND NOTATIONS

The Fourier-Laguerre series of a function $f(x) \in L[0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \qquad ...(1.1)$$

where

$$\Gamma(\alpha + 1)\binom{n+\alpha}{n} a_n = \int_0^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha)}(y) f(y)\, dy \qquad ...(1.2)$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomials of order $\alpha > -1$.

The nth Cesàro mean $\sigma_n^k(f; 0)$ of order k of the series (1.1) at the point $x = 0$ is given by (see Szegö 1959, p. 269)

$$\sigma_n^k(f; 0) = \{A_n^k \Gamma(\alpha + 1)\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha+k+1)}(y) f(y)\, dy \qquad ...(1.3)$$

where

$$A_n^k = \binom{n+k}{n} \sim \frac{n^k}{\Gamma(k+1)}. \qquad ...(1.4)$$

Generalizing a result of Flett (1956) on Fourier-trigonometric series, Siddiqi (1971), in the same direction, proved the following theorem :

Theorem A — For $0 < k < 1$ and $0 < \delta \leqslant \pi$

$$\sigma_n^k(f; x) - f(x) = O\left(\psi\left(\frac{1}{n}\right)\right) + O(n^{-k})$$

if x is a point such that

$$\int_0^t |\, d\phi(u) \,| \leqslant A\,\psi(t). \quad 0 \leqslant t \leqslant \delta$$

where $\psi(t)$ is a positive increasing function such that

$$\int_{1/n}^{\delta} \frac{\psi(t)}{t^2}\, dt = O\left(n\psi\left(\frac{1}{n}\right)\right), \quad n \to \infty$$

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

and A is a constant.

We propose to extend the above result to Laguerre series at the point $x = 0$. To achieve this end we shall use the Stieltjes integrals adopting a technique different from that of Hille (1926) in the convergence theory of Laguerre series. Finally, we arrive at a deeper insight into the behaviour of the order of approximation by Cesàro means than that of Gupta (1971). We write

$$\phi(y) = f(y) - f(0).$$

2. Theorem

We establish the following theorem

Theorem — For $k > \alpha + \frac{1}{2}$, $\alpha > -1$

$$\sigma_n^k(f;\, 0) = f(0) + O(n^{-(1/2)}) + O\left(\psi\left(\frac{1}{n}\right)\right), \text{ provided that}$$

$$\int_0^t |\, d\phi(u) \,| \leqslant A\psi(t), \quad 0 \leqslant t \leqslant \omega < \infty \qquad \text{...(2.1)}$$

$$\int_\omega^\infty e^{-(y/2)} y^{-(1/3)} \,|\, d\phi(y) \,| < \infty \qquad \text{...(2.2)}$$

$$\int_\omega^\infty e^{-(y/2)} y^{-(4/3)} \,|\, \phi(y) \,|\, dy < \infty \qquad \text{...(2.3)}$$

where $\psi(t)$ is a positive increasing function such that

$$\int_{1/n}^{\delta} \frac{\psi(t)}{t^2}\, dt = o\left(\sqrt{n}\,\psi\left(\frac{1}{n}\right)\right), \quad n \to \infty \qquad \text{...(2.4)}$$

A being a constant not necessarily the same at each occurrence.

3. Order Estimates and Asymptotic Properties

The following order estimates and asymptotic properties of Laguerre polynomials (see Szegö 1959, pp. 175 and 239) will be required in the proof.

Let α be arbitrary and real, c and ω fixed positive constants, and let $n \to \infty$, then

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-(2\alpha+1)/4}\, O(n^{(2\alpha-1)/4}), & \frac{c}{n} \leqslant x \leqslant \omega \\ O(n^{\alpha}), & 0 \leqslant x \leqslant \frac{c}{n}. \end{cases} \qquad \ldots(3.1)$$

If α and λ be arbitrary and real, $\omega > 0$, $0 < \eta < 4$, then for $n \to \infty$

$$\max e^{-x/2} x^{\lambda} \mid L_n^{(\alpha)}(x) \mid \sim n^{Q} \qquad \ldots(3.2)$$

where

$$Q = \begin{cases} \max\left(\lambda - \tfrac{1}{2}, \tfrac{\alpha}{2} - \tfrac{1}{4}\right), & \omega \leqslant x \leqslant (4-\eta)\, n \\ \max\left(\lambda - \tfrac{1}{3}, \tfrac{\alpha}{2} - \tfrac{1}{4}\right), & x \geqslant \omega \end{cases} \qquad \ldots(3.3)$$

the maxima being taken in the intervals pointed out in the right-hand members of (3.3).

4. Proof of the Theorem

From (1.3) and using orthogonal property of Laguerre polynomials, we have

$$\sigma_n^k(f; 0) - f(0) = \{A_n^k \Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^{\alpha} L_n^{(\alpha+k+1)}(y)\, \phi(y)\, dy$$

$$= -\{n A_n^k \Gamma(\alpha+1)\}^{-1} \int_0^\infty D[e^{-y} y^{\alpha+k+2} D L_n^{(\alpha+k+1)}(y)] \frac{\phi(y)}{y^{k+1}}\, dy$$

by the relation (Rainville 1960, p. 204)

$$D[e^{-y} y^{\alpha+1} D L_n^{(\alpha)}(y)] + n e^{-y} y^{\alpha} L_n^{(\alpha)}(y) = 0, \; D = \frac{d}{dx}$$

$$= -\{n A_n^k \Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^{\alpha+1} L_{n-1}^{(\alpha+k+2)}(y)\, d\phi(y)$$

$$+ \{n A_n^k \Gamma(\alpha+1)\}^{-1} (k+1) \int_0^\infty e^{-y} y^{\alpha} L_{n-1}^{(\alpha+k+2)}(y)\, \phi(y)\, dy$$

by the relation $\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)$

$$= M + N, \text{ say.} \qquad \ldots(4.1)$$

We write

$$M = \int_0^{1/n} + \int_{1/n}^{\omega} + \int_{\omega}^{\infty} = I_1 + I_2 + I_3. \qquad \ldots(4.2)$$

Now using (3.1) and (2.1), we have

$$\max_{0 \leqslant y \leqslant 1/n} | I_1 | = O(n^{-k-1})\, O(n^{\alpha+k+2}) \int_0^{1/n} e^{-y} y^{\alpha+1} \, | d\phi(y) |$$

$$= O\left\{\psi\left(\frac{1}{n}\right)\right\} \qquad \ldots(4.3)$$

also

$$\max_{1/n \leqslant y \leqslant \omega} | I_2 | = O(n^{(2\alpha-2k-1)/4}) \int_{1/n}^{\omega} y^{(2\alpha-2k-1)/4} \, | d\phi(y) |$$

$$= O(n^{-1/2}) \int_{1/n}^{\omega} y^{-1/2} \, | d\phi(y) |$$

$$= O(n^{-1/2}) \left[\{y^{-1/2}\psi(y)\}_{1/n}^{\omega} + \tfrac{1}{2} \int_{1/n}^{\omega} y^{-3/2}\psi(y)\, dy\right]$$

$$= O(n^{-1/2}) + O\left\{\psi\left(\frac{1}{n}\right)\right\} + O(n^{-1/2}) \int_{1/n}^{\omega} \frac{\psi(y)}{y^2}\, dy$$

$$= O(n^{-1/2}) + O\left\{\psi\left(\frac{1}{n}\right)\right\}. \qquad \ldots(4.4)$$

Putting

$$\lambda - \tfrac{1}{3} = \frac{\alpha + k + 2}{2} - \tfrac{1}{4} = \left(\frac{\alpha}{2} + \tfrac{1}{4}\right) + \frac{k}{2} + \tfrac{1}{2} \leqslant k + \tfrac{1}{2}$$

in (3.2) and (3.3), and using (2.2), we have

$$\max_{\omega \leqslant y < \infty} | I_3 | = O(n^{-k-1}) \int_{\omega}^{\infty} e^{-y/2} y^{\alpha+1} n^{(2k+1)/2} y^{-(6k+5)/6} \, | d\phi(y) |$$

$$= O(n^{-1/2}) \int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha-6k+1)/6} \, | d\phi(y) |$$

$$= O(n^{-1/2}) \int_{\omega}^{\infty} e^{-y/2} y^{-1/3} \, | d\phi(y) |$$

$$= O(n^{-1/2}). \qquad \ldots(4.5)$$

Again we write

$$N = \int_0^{1/n} + \int_{1/n}^{\omega} + \int_{\omega}^{\infty} = J_1 + J_2 + J_3. \qquad \ldots(4.6)$$

We see that $\phi(0) = 0$, so that

$$| \phi(y) | \leqslant \int_0^y | d\phi(t) | \leqslant A\psi(y)$$

and, therefore, using (3.1), we have

$$\max_{0 \leqslant y \leqslant 1/n} | J_1 | = O(n^{\alpha+1}) \int_0^{1/n} y^{\alpha}\psi(y)\, dy$$

$$= O(n^{\alpha+1}) \psi\left(\frac{1}{n}\right) \int_0^{1/n} y^{\alpha} dy = O\left(\psi\left(\frac{1}{n}\right)\right) \qquad \ldots(4.7)$$

and

$$\max_{1/n \leqslant y \leqslant \omega} | J_2 | = O(n^{(2\alpha-2k-1)/4}) \int_{1/n}^{\omega} y^{(2\alpha-2k-5)/4} | \phi(y) | \, dy$$

$$= O(n^{(2\alpha-2k-1)/4}) \int_{1/n}^{\omega} y^{(2\alpha-2k+1)/4}\, y^{-3/2} | \phi(y) | \, dy$$

$$= O(n^{-1/2}) \int_{1/n}^{\omega} \frac{\psi(y)}{y^2} dy$$

$$= o\left\{\psi\left(\frac{1}{n}\right)\right\}. \qquad \ldots(4.8)$$

Finally, proceeding as in I_3, so that $(3\lambda - 1)/3 = (2k + 1)/2$, and using (2.3), we obtain

$$\max_{\omega \leqslant y < \infty} | J_3 | = O(n^{-1/2}) \int_{\omega}^{\infty} e^{-y/2}\, y^{(6\alpha-6k-5)/6} | \phi(y) | \, dy$$

$$= O(n^{-1/2}) \int_{\omega}^{\infty} e^{-y/2}\, y^{(2\alpha-2k+1)/2}\, y^{-4/3} | \phi(y) | \, dy$$

$$= O(n^{-1/2}) \int_{\omega}^{\infty} e^{-y/2}\, y^{-4/3} | \phi(y) | \, dy$$

$$= O(n^{-1/2}). \qquad \ldots(4.9)$$

This completes the proof of the theorem.

If we put $\psi(t) = t^{\beta}$, $0 < \beta \leqslant \frac{1}{2}$, we have the following Corollary analogous to a result of Flett (1956) on Fourier-trigonometric series.

Corollary — For $k > \alpha + \frac{1}{2}$, $\alpha > -1$ and for $0 < \beta \leqslant \frac{1}{2}$

$$\sigma_n^k (f; 0) = f(0) + O(n^{-\beta})$$

provided that the conditions from (2.1) to (2.3) are satisfied.

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