A USE FOR A DERIVATIVE OF COMPLEX ORDER IN THE FRACTIONAL CALCULUS

OR

WHAT, INDEED, IS $d^{3-(1/2)i}/dx^{3-(1/2)i}$ AND WHAT CAN YOU DO WITH IT ?

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The purpose of this paper is not to present a new result; its purpose is to present an alternative technique in which an explicit use is made of a derivative of complex order. The Riemann-Liouville operator of arbitrary order ${}_0D_x^{\nu}$ is applied to the Euler type differential equation $x^2y'' + \alpha xy' + \beta y = 0$. The result is the same as that obtained by standard classical means but the method of fractional calculus is used. Introductory expository material is included.

INTRODUCTION

Liouville (1832) and Riemann (1876) developed logical definitions of fractional operations. The definite integral, called the Riemann-Liouville integral, for integration of arbitrary order is

$$ {}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x - t)^{\nu-1} f(t)\, dt \qquad ...(1)$$

where $\mathrm{Re}(\nu) > 0$ to ensure convergence of the integral. When $c = 0$ we have Riemann's definition and when $c = -\infty$ we have Liouville's definition. The function f is such that on $[c, x]$ the integral converges. For values of $\nu < 0$ the above integral would, in general, diverge but because ${}_cD_x^{-\nu} f(x)$ can be shown to be an analytic function of ν in the region of convergence $\mathrm{Re}(\nu) > 0$, it can be defined outside this region of convergence as the analytic continuation of this function. This is the same procedure as that used to define $\Gamma(z)$ outside the region of convergence of the integral $\int_0^{\infty} e^{-t}t^{z-1}dt$.

In the context of the fractional calculus a function of the type t^a, Re $(a) > -1$, is said to belong to the Riemann class of functions because for $c \geqq 0$, the integral (1) converges. Now, if $f(t)$ depends upon a parameter a, we write $f(t, a)$ instead of $f(t)$. Let $f(t, a)$ be continuous in both t and a and analytic in a for each t in $c \leqq t \leqq x$, for $a \in R$, where R is some complex region of the a-plane, and let $f(t, a)$ be such that the integral (1) converges for each $a \in R$. Then if the integral (1) written as the limit

$$\lim_{\delta \to 0} \int_{c+\delta}^{x} (x - t)^{\nu-1} f(t, a)\, dt$$

is uniformly convergent in a over $a \in R$, then it can be shown that ${}_cD_x^{-\nu} f(x, a)$ is analytic in ν and a over its region of convergence. For example, we can define ${}_cD_x^{3/4}\, x^{-3/2}$ by analytic continuation even though the integral $\int_c^x (x - t)^{-7/4}\, t^{-3/2}\, dt$ is divergent.

The notation ${}_cD_x^{-\nu}$ which denotes the operator of integration of arbitrary order and the corresponding operator ${}_cD_x^{\nu}$ which denotes differentiation of arbitrary order was invented by Davis (1936). The subscripts adjoining D are the terminals of integration and are vital in applications to avoid ambiguities. Differentiation of arbitrary order is given by

$${}_cD_x^{\nu} f(x) = {}_cD_x^{m-p} f(x) = {}_cD_x^{m}\, {}_cD_x^{-p} f(x)$$

$$= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(p)} \int_c^x (x - t)^{p-1} f(t)\, dt \right\} \qquad \ldots(2)$$

where (for convenience) m is taken to be the least integer greater than Re (ν), $\nu = m - p$, $0 < p \leqq 1$, and ${}_cD_x^{m}$ is the ordinary differentiation operator d^m/dx^m. For a wide class of functions f, the integral above is a beta integral and is readily evaluated.

For functions of the type x^a, Re $(a) > -1$ and Re $(\nu) > 0$, we have for integration and differentiation of arbitrary order

$${}_0D_x^{-\nu} x^a = \frac{\Gamma(a + 1)}{\Gamma(a + \nu + 1)}\, x^{a+\nu} \qquad \ldots(3)$$

$${}_0D_x^{\nu} x^a = \frac{\Gamma(a + 1)}{\Gamma(a - \nu + 1)}\, x^{a-\nu}. \qquad \ldots(4)$$

It is not necessary for the application to be presented in this paper to delve into all the precise conditions under which the above formulas can be used. Interested readers can pursue these details in a paper by Northover (1977). It suffices for our purpose to state that formulas (3) and (4) hold true by analytic continuation for all values of a and ν if the functions in which these parameters appear are analytic. For example, a cannot be a negative integer in (3) unless $(a + \nu)$ is also a negative integer, and a cannot be a negative integer in (4) unless $(a - \nu)$ is also a negative integer.

Some additional developments will be useful to make the reading of this paper more accessible to the general reader. The Leibniz rule for the nth derivative of a product can be generalized to a derivative of arbitrary order:

$$ {}_cD_x^\nu f(x)\, g(x) = \sum_{n=0}^{\infty} \binom{\nu}{n} {}_cD_x^{(n)} f(x)\, {}_cD_x^{(\nu-n)} g(x) \qquad ...(5)$$

where ${}_cD_x^{(n)}$ is the ordinary differentiation operator d^n/dx^n and $D^{(\nu-n)}$ is a fractional operation given by formulas (3) and (4). $\binom{\nu}{n}$ is the generalized binomial coefficient $\Gamma(\nu + 1)/ n!\, \Gamma(\nu - n + 1)$. It is often convenient to choose one of the factors in the product $f(x)\, g(x)$ so that the series will terminate after a number of ordinary differentiations. For example, let $f(x) = x$, then for the derivative of order $\frac{1}{2}$ of the product $xg(x)$ the series terminates for $n = 2$:

$$ {}_0D_x^{1/2}\, xg(x) = x\, {}_0D_x^{1/2}\, g(x) + \tfrac{1}{2}\, {}_0D_x^{-1/2}\, g(x). \qquad ...(6)$$

For an explicit $g(x)$ one can use formulas (3) and (4) for integration and differentiation of order $\frac{1}{2}$.

The following desiderata (Ross 1975) is tabulated because they will be referred to in the application given later. Subscripts are omitted for convenience.

The operation of order zero leaves the operation unchanged:

$$D^0 f = f \qquad ...(7a)$$

Fractional operators are linear:

$$D^{-\nu}(\alpha f + \beta g) = \alpha D^{-\nu} f + \beta D^{-\nu} g \qquad ...(7b)$$

The law of exponents (indices) for integration of arbitrary order is

$$D^{-\mu} D^{-\nu} f(x) = D^{-\mu-\nu} f(x) \qquad ...(7c)$$

where $\operatorname{Re}(\mu) > 0$ and $\operatorname{Re}(\nu) > 0$.

The law of exponents for differentiation of arbitrary order for general ν and n a positive integer is

$$ {}_cD_x^{\nu}\,{}_cD_x^{n}\,f(x) = {}_cD_x^{\nu+n} f(x) - \sum_{k=0}^{n-1} \frac{(x-c)^{-\nu-n+k}}{\Gamma(-\nu-n+k+1)} f^{(k)}(c). \qquad \ldots(7d) $$

The above result is obtained from (1) by replacing ν with $\nu + n$. The integral for $D^{\nu+n}f$, with appropriate restrictions on ν to ensure convergence, is integrated by parts. The truth of (7*d*) for general ν follows by analytic continuation.

Application

The purpose of this paper is to show how a derivative of complex order can be used to solve the differential equation

$$ x^2y'' - 3xy' + \left(\frac{13}{4} + i\right) y = 0. \qquad \ldots(8) $$

The idea of solving certain differential equations by means of the fractional calculus started first with Liouville and was later augmented by Holmgren (1867). The fractional operations of the early mathematicians were formalized and not justified. For example, they formally wrote $D^{\nu}D^2 f = D^{\nu+2} f$ and omitted the additional terms given in (7*d*). That omission would be alright if the restriction were imposed that the function f vanishes to a sufficiently high order at the lower terminal of integration, namely $f^{(k)}(c) = 0$. The work which follows broadens these concepts because no such restriction is imposed.

We consider the general case

$$ x^2y'' + \alpha xy' + \beta y = 0 \qquad \ldots(9) $$

where α and β are real or complex constants. Operate on each term of (9) with ${}_cD_x^{\nu}$. ν will depend on α and β. Subscripts will be omitted for convenience. We have

$$ D^{\nu}(x^2y'') + \alpha D^{\nu}(xy') + \beta D^{\nu}y = 0. \qquad \ldots(10) $$

Apply the Leibniz rule (5) to the first and second terms of the above and we get

$$ D^{\nu}(x^2y'') = x^2D^{\nu}y'' + 2\nu xD^{\nu-1}y'' + \nu(\nu-1)\,D^{\nu-2}y'' \qquad \ldots(11) $$

and

$$ \alpha D^{\nu}(xy') = \alpha xD^{\nu}y' + \alpha\nu D^{\nu-1}y'. \qquad \ldots(12) $$

We observe that y'' is D^2y. So, by (7*d*) we can write

$$ \begin{aligned} D^{\nu}D^2y &= D^{\nu+2}y - \sum_{k=0}^{1} \frac{(x-c)^{-\nu-2+k}}{\Gamma(-\nu-1+k)} f^{(k)}(c) \\ &= D^{\nu+2}y - \frac{(x-c)^{-\nu-2}}{\Gamma(-\nu-1)} y(c) - \frac{(x-c)^{-\nu-1}}{\Gamma(-\nu)} y'(c). \end{aligned} \qquad \ldots(13) $$

In the same way the following results are obtained :

$$D^{\nu}y' = D^{\nu+1}y - \frac{(x-c)^{-\nu-1}}{\Gamma(-\nu)}\,y(c) \qquad \text{...(14)}$$

$$D^{\nu-1}y'' = D^{\nu+1}y - \frac{(x-c)^{-\nu-1}}{\Gamma(-\nu)}\,y(c) - \frac{(x-c)^{-\nu}}{\Gamma(-\nu+1)}\,y'(c) \qquad \text{...(15)}$$

$$D^{\nu-2}y'' = D^{\nu}y - \frac{(x-c)^{-\nu}}{\Gamma(-\nu+1)}\,y(c) - \frac{(x-c)^{-\nu+1}}{\Gamma(-\nu+2)}\,y'(c) \qquad \text{...(16)}$$

$$D^{\nu-1}y' = D^{\nu}y - \frac{(x-c)^{-\nu}}{\Gamma(-\nu+1)}\,y(c). \qquad \text{...(17)}$$

Then, after some reductions, using well-known properties of the gamma function, we find on the substitution of (13) through (17) into eqn. (10) that we now have

$$x^2D^{\nu+2}y + (\alpha + 2\nu)\,xD^{\nu+1}y + (\alpha\nu + \nu(\nu-1) + \beta)\,D^{\nu}y = F(c, x) \qquad \text{...(18)}$$

where

$$F(c, x) = \frac{c(x-c)^{-\nu-2}}{\Gamma(-\nu)}\{c(x-c)\,y'(c) + (x(\alpha-2) - c(\alpha+\nu-1))\,y(c)\}. \qquad \text{...(19)}$$

We consider the Riemann case ($c = 0$). Assuming that $F(c, x) \to 0$ as $c \to 0$, (18) becomes

$$x^2D^{\nu+2}y + (\alpha + 2\nu)\,xD^{\nu+1}y + (\alpha\nu + \nu(\nu-1) + \beta)\,D^{\nu}y = 0. \qquad \text{...(20)}$$

A solution can be obtained if the coefficient of $D^{\nu}y$ is set equal to zero. The resulting quadratic equation

$$\nu^2 + (\alpha - 1)\,\nu + \beta = 0$$

yields

$$\nu_1 = \frac{1-\alpha}{2} + \sqrt{\left(\frac{1-\alpha}{2}\right)^2 - \beta} \quad \text{and} \quad \nu_2 = \frac{1-\alpha}{2} - \sqrt{\left(\frac{1-\alpha}{2}\right)^2 - \beta}. \qquad \text{...(21}a\text{)}$$

and it follows that

$$\nu_1 + \nu_2 = 1 - \alpha. \qquad \text{...(21}b\text{)}$$

Now, let $D^{\nu+1}y = u$, so that $D^{\nu+2}y = u'$. Equation (20) is then

$$x^2u' + (\alpha + 2\nu)\,xu = 0 \qquad \text{...(22)}$$

which has the solution

$$u = kx^{-(\alpha+2\nu)} \qquad \text{...(23)}$$

that is,

$$ {}_0D_x^{\nu+1}\, y = kx^{-\alpha-2\nu} \qquad \text{...(24)} $$

where k is a constant of integration.

To obtain y we operate on both sides of (24) with $D^{-\nu-1}$. We observe that $\mathrm{Re}\,(-\nu-1)$ and $\mathrm{Re}\,(\nu+1)$ cannot both be negative. Thus, the index law (7*c*) cannot be used to deduce $D^{-\nu-1}(D^{\nu+1}y) = D^0y = y$. What is needed is a rule which determines conditions for the validity of the relation $D^p(D^q f) = D^{p+q} f$ when $\mathrm{Re}\,(p)$ and $\mathrm{Re}\,(q)$ are no longer both negative. An investigation can be found in a paper by Northover (1977). In this particular application, the important condition is that $\alpha + 2\nu$ must not be a positive integer unless ν is then a negative integer. Subject to this condition we obtain

$$ y = k\, {}_0D_x^{-(\nu+1)}\, x^{-(\alpha+2\nu)}. \qquad \text{...(25)} $$

The integral (1) would, in general, diverge with $t^{-(\alpha+2\nu)}$ but, as stated earlier, meaning can be given to the integral by analytic continuation. Appeal is made for the formal use of formula (3) where $a = -\alpha - 2\nu$:

$$ y = k\,\frac{\Gamma(-\alpha-2\nu+1)}{\Gamma(-\alpha-\nu+2)}\, x^{-\alpha-\nu+1} \qquad \text{...(26)} $$

where $\alpha + 2\nu$ is not a positive integer unless ν is then a negative integer. This is accord with the remark made after formula (4).

Denote the complex coefficient above as K_1. From (26) $y(c)$ and $y'(c)$ can be computed. Then $F(c, x)$ in (19) reduces to

$$ F(c, x) = K_1\,\frac{x(x-c)^{-\nu-2}c^{2-\alpha-\nu}}{\Gamma(-\nu-1)}. \qquad \text{...(27)} $$

Thus,

$$ F(c, x) \to 0 \ \text{ as } \ c \to 0 $$

if and only if

$$ \mathrm{Re}\,(\alpha+\nu) < 2. \qquad \text{...(28)} $$

We now consider eqn. (8) :

$$ x^2y'' - 3xy' + \left(\frac{13}{4} + i\right)y = 0. $$

Here $\alpha = -3$ and $\beta = \dfrac{13}{4} + i$. From (21*a*) and (21*b*) we obtain

$$ \nu_1 = 3 - \tfrac{1}{2}i \ \text{ and } \ \nu_2 = 1 + \tfrac{1}{2}i. $$

The solution of eqn. (8) as given by (26) is the sum of two particular solutions :

$$ y = K_1x^{1+(1/2)i} + K_2x^{3-(1/2)i} \qquad \text{...(29)} $$

where the K's are complex constants. This result is the same as that obtained by classical methods. It is important to note that the inequality (28) is satisfied by α and by both ν_1 and ν_2.

The usual example in text books is

$$x^2y'' + xy' + y = 0. \qquad \ldots(30)$$

Here $\alpha = \beta = 1$. Then $\nu_1 = i$ and $\nu_2 = -i$, and inequality (28) is satisfied by α and by both ν_1 and ν_2. The solution as given by (26) is the sum of the two particular solutions :

$$y = K_1 x^i + K_2 x^{-i} \qquad \ldots(31)$$

which is often written as

$$y = C_1 \cos(\ln x) + C_2 \sin(\ln x).$$

Concluding Comments

The operators ${}_0D_x^{3-(1/2)i}$ and ${}_0D_x^{1+(1/2)i}$ can be applied directly to eqn. (8) to get the desired solution and the operators of pure imaginary order D^i and D^{-i} can be applied directly to eqn. (30). The use of these operators is justified because inequality (28), Re $(\alpha + \nu) < 2$, is satisfied. When Re $(\alpha + 2\nu) > 0$, it can be shown that the Liouville operator ${}_{-\infty}D_x^{\nu}$ can be used. Equation (30) is an instance where both inequalities are satisfied and either the Riemann or Liouville operator of complex order may be used.

In the history of the attempts to solve certain ordinary differential equations by fractional operators, there are no examples of the use of a derivative of complex order and there are no formulations of inequalities that would justify the use of the Riemann or Liouville operator.

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