ON THE (N, p_n^α) SUMMABILITY OF FOURIER SERIES AND ITS ALLIED SERIES

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The Nörlund summability of Fourier series and its allied series has been studied by various authors under various types of criteria and conditions. In this paper, a different criterion for the (N, p_n^α) summability of Fourier series and its allied series has been studied. Some results which may be considered as analogous to a theorem of Tripathi and Prasad (1973), are presented.

§1. If $\sum_{n=0}^{\infty} a_n$ is a series, we shall use the notation

$$S_n = \sum_{r=0}^{n} a_r. \qquad \ldots(1.1)$$

For α real, define

$$\epsilon_0^\alpha = 1,$$

$$\epsilon_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{\lfloor n}, \; (n = 1, 2, 3, \ldots).$$

Let $\{p_n\}$ be a sequence with $p_0 > 0$ and $p_n \geqslant 0$ for $n > 0$. We define

$$p_n^\alpha = \sum_{r=0}^{n} \epsilon_{n-r}^{\alpha-1} \, p_r.$$

We write

$$P_n^{(\alpha)} = \sum_{r=0}^{n} p_r^\alpha \qquad \ldots(1.2)$$

where

$$P_n = \sum_{r=0}^{n} p_r. \qquad \ldots(1.3)$$

Definition (Cass 1969) — Nörlund summability (N, p_n^α).

For $\alpha > -1$ and $\sum_{r=0}^{\infty} a_r$, a series, let

$$t_n^{(\alpha)} = \frac{1}{P_n^{(\alpha)}} \sum_{r=0}^{n} p_{n-r}^{\alpha} S_r. \qquad \ldots(1.4)$$

If $t_n^{(\alpha)} \to S$ as $n \to \infty$, we write

$$\sum_{r=0}^{\infty} a_r = S(N, p_n^\alpha) \text{ or } S_n \to S(N, p_n^\alpha).$$

§2. We shall consider a function $f(t)$ of bounded variation, integrable in the sense of Lebesgue and periodic with period 2π.

If
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx,$$

then $f(t)$ generates the Fourier-Lebesgue series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \qquad \ldots(2.1)$$

The series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} n\, B_n(t) \qquad \ldots(2.2)$$

which is obtained by differentiating (2.1) term by term, is called the first derived series or the derived Fourier series of $f(t)$.

The conjugate series of (2.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t). \qquad \ldots(2.3)$$

The series

$$\sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt) \qquad \ldots(2.4)$$

is called the derived conjugate series of the Fourier series. We write

$$\varphi(t) = \varphi(x, t) = f(x+t) + f(x-t) - 2f(x)$$

$$\Phi(t) = \int_0^t |\varphi(u)| \, du$$

$$g(t) = g(x, t) = f(x+t) - f(x-t) - 2tf'(x)$$

$$G(t) = \int_0^t | dg(u) |$$

$$\psi(t) = \psi(x, t) = f(x+t) - f(x-t)$$

$$\Psi(t) = \int_0^t |\psi(u)| \, du$$

$$\tau = \left[\frac{1}{t}\right], \text{ where } [t] \text{ denotes the integral part of } t.$$

$$N_n(t) = \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \frac{\sin (n-k+\frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$\bar{N}_n(t) = \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \frac{\cos (n-k+\frac{1}{2})t}{\sin \frac{1}{2}t}.$$

§3. Tripathi and Prasad (1973) proved the following theorem for the (N, p_n^α) summability of the derived conjugate series of a Fourier series.

Theorem A — A Nörlund method of summation (N, p_n^α), defined by a real, non-negative, non-increasing sequence of coefficients $\{p_n^\alpha\}$ such that $P_n^{(\alpha)} \to \infty$ as $n \to \infty$, sums the derived conjugate series of the Fourier series of the function $f(x)$ to the sum

$$-\frac{1}{4\pi} \int_0^\pi h(t) \operatorname{cosec}^2 \tfrac{1}{2}t \, dt$$

at every point x at which this integral exists and

$$\psi(t) = o\left(\frac{p_\tau^\alpha}{P_\tau^{(\alpha)}}\right), \quad t \to +0$$

where

$$h(t) = f(x + t) + f(x - t) - 2f(x)$$

and

$$\psi(t) = \int_0^t | dh(u) | .$$

The object of the present paper is to establish some results analogous to the above theorem for the Fourier series, derived Fourier series and conjugate Fourier series. Thus, we prove the following theorems :

Theorem 1 — The Fourier series of the function $f(t)$ is summable (N, p_n^α) to the sum $f(x)$ at the point $t = x$ where

$$\Phi(t) \equiv \int_0^t | \varphi(u) | du = o\left(\frac{p_\tau^\alpha}{P_\tau^{(\alpha)}}\right), \quad t \to +0$$

and $\{p_n^\alpha\}$ is a real, non-negative, non-increasing sequence of coefficients such that

$$P_n^{(\alpha)} \to \infty \text{ as } n \to \infty.$$

Theorem 2 — The derived Fourier series (2.2) is summable by (N, p_n^α) method to the sum $f'(x)$, at the point $t = x$, at which

$$G(t) \equiv \int_0^t | dg(u) | = o\left(\frac{p_\tau^\alpha}{P_\tau^{(\alpha)}}\right), \quad t \to +0$$

where $\{p_n^\alpha\}$ is the same as in Theorem 1 and $f'(x)$ denotes the first generalized differential coefficient of $f(x)$.

Theorem 3 — The conjugate series (2.3) of the Fourier series is summable by (N, p_n^α) method to the sum

$$\frac{1}{2\pi} \int_0^\pi \psi(t) \cot \tfrac{1}{2}t \, dt$$

at the point $t = x$ at which this integral exists and

$$\Psi(t) \equiv \int_0^t |\psi(u)|\, du = O\left(\frac{p_\tau^\alpha}{P_\tau^{(\alpha)}}\right), \quad t \to +0$$

where $\{p_n^\alpha\}$ is the same as in Theorem 1.

§4. The following lemmas are required for the proof of the above theorems :

Lemma 1 — If $\{p_n^\alpha\}$ is non-negative and non-increasing, then for $0 \leqslant a < b \leqslant \infty$ and $0 \leqslant t \leqslant \pi$ and any n

$$\left| \sum_{k=a}^{\pi} p_k^\alpha\, e^{i(n-k)t} \right| \leqslant A P_\tau^{(\alpha)}$$

where A is an absolute constant.

The proof of the lemma follows on the lines of McFadden (1942).

As a consequence of this lemma, we have

$$N_n(t) = \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \frac{\sin (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$= O\left(\frac{P_\tau^{(\alpha)}}{t P_n^{(\alpha)}}\right)$$

and

$$\bar{N}_n(t) = \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \frac{\cos (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$= O\left(\frac{P_\tau^{(\alpha)}}{t P_n^{(\alpha)}}\right).$$

Lemma 2 — For $0 \leqslant t \leqslant \dfrac{1}{n}$

$$N_n(t) = O(n).$$

PROOF :

$$N_n(t) = \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha} \frac{\sin (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha} \frac{(2n - 2k + 1) \mid \sin \frac{1}{2}t \mid}{\mid \sin \frac{1}{2}t \mid}\right]$$

$$= O\left[\frac{n}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha}\right] = O(n).$$

§5. PROOF OF THEOREM 1 — We have to show, under the hypothesis of the theorem, that

$$\int_0^{\pi} \varphi(t) . N_n(t)\, dt = o(1), \text{ as } n \to \infty.$$

We set, for $0 < \delta < \pi$,

$$\int_0^{\pi} \varphi(t) . N_n(t)\, dt = [\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}]\, \varphi(t) . N_n(t)\, dt$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

Now

$$I_1 = O[n \int_0^{1/n} \mid \varphi(t) \mid dt], \text{ by Lemma 2}$$

$$= o\left[\frac{n p_n^{\alpha}}{P_n^{(\alpha)}}\right]$$

$$= o(1), \text{ as } n \to \infty \text{ (since } np_n^{\alpha} \leqslant P_n^{(\alpha)}).$$

$$I_2 = O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} \mid \varphi(t) \mid \cdot \frac{P_\tau^{(\alpha)}}{t}\, dt\right], \text{ by Lemma 1}$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \left\{\Phi(t) \frac{P_\tau^{(\alpha)}}{t}\right\}_{1/n}^{\delta}\right]$$

$$+ O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} \Phi(t) \frac{P_\tau^{(\alpha)}}{t^2} dt\right]$$

$$+ O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} \Phi(t) \frac{1}{t} dP_\tau^{(\alpha)}\right]$$

$$= I_{2\cdot1} + I_{2\cdot2} + I_{2\cdot3}, \text{ say.}$$

$$I_{2\cdot1} = O\left[\frac{1}{P_n^{(\alpha)}} \left\{\frac{p_\tau^{\alpha}}{t}\right\}_{1/n}^{\delta}\right]$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \frac{p_{(1/\delta)}^{\alpha}}{\delta}\right] + O\left[\frac{1}{P_n^{(\alpha)}} \frac{p_n^{\alpha}}{\frac{1}{n}}\right]$$

$$= o(1) + o(1) = o(1), \text{ as } n \to \infty.$$

$$I_{2\cdot2} = O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} \Phi(t) \frac{P_\tau^{(\alpha)}}{t^2} dt\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} \frac{p_\tau^{\alpha}}{t^2} dt\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}} \int_{1/\delta}^{n} p_{[s]}^{\alpha} ds\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha}\right] = o(1), \text{ as } n \to \infty.$$

$$I_{2\cdot3} = O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} \Phi(t) \frac{dP_\tau^{(\alpha)}}{t}\right]$$

(equation continued on p. 54)

$$= O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/\delta}^{n} \Phi\left(\frac{1}{u}\right) u \, dP_{[u]}^{(\alpha)}\right]$$

$$= o(1) + O\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=1}^{n-1} kP_k^{\alpha} \, \Phi\left(\frac{1}{k}\right)\right]$$

$$= o(1) + O\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=1}^{n-1} P_k^{(\alpha)} \Phi\left(\frac{1}{k}\right)\right]$$

$$= o(1) + o\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=1}^{n-1} P_k^{(\alpha)} \frac{p_k^{\alpha}}{P_k^{(\alpha)}}\right]$$

$$= o(1) + o\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=1}^{n-1} p_k^{\alpha}\right]$$

$$= o(1), \text{ as } n \to \infty.$$

Thus, $\quad I_2 = o(1).$

Finally, since the method of summability is regular, we have $I_3 = o(1)$, as $n \to \infty$ by Riemann-Lebesgue theorem.

This proves Theorem 1.

PROOF OF THEOREM 2 — Denoting by $S_n(x)$ the sum of the first n terms of the series (2.2) at the point $t = x$, we get

$$S_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \left\{\frac{d}{dx} \frac{\sin (n + \frac{1}{2}) (x - u)}{\sin \frac{1}{2}(x - u)}\right\} f(u) \, du$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} f(u) \left\{\frac{d}{du} \frac{\sin (n + \frac{1}{2}) (x - u)}{\sin \frac{1}{2}(x - u)}\right\} du$$

$$= -\frac{1}{2\pi} \int_0^{\pi} \{f(x + t) - f(x - t)\} \left\{\frac{d}{dt} \frac{\sin (n + \frac{1}{2})t}{\sin \frac{1}{2}t}\right\} dt.$$

Now integrating by parts the right hand side of the above, we get

$$S_n(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin (n + \frac{1}{2})\, t}{\sin \frac{1}{2}\, t}\, d\{f(x+t) - f(x-t)\}$$

$$= \frac{1}{2\pi} \int_0^\pi \frac{\sin (n + \frac{1}{2})t}{\sin \frac{1}{2}t}\, dg(t) + f'(x).$$

Therefore,

$$S_n(x) - f'(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin (n + \frac{1}{2})\, t}{\sin \frac{1}{2}\, t}\, dg(t) \text{ (Zygmund 1935).}$$

Hence, we obtain

$$\frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \{S_{n-k}(x) - f'(x)\}$$

$$= \frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \cdot \frac{1}{2\pi} \int_0^\pi \frac{\sin (n - k + \frac{1}{2})\, t}{\sin \frac{1}{2}\, t}\, dg(t)$$

$$= \int_0^\pi dg(t) \cdot \frac{1}{2\pi P_n^{(\alpha)}} \left\{ \sum_{k=0}^{n} p_k^\alpha \frac{\sin (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t} \right\}$$

$$= \int_0^\pi dg(t) \,.\, N_n(t).$$

To prove the theorem, we have to show, under the hypothesis of the theorem, that

$$\int_0^\pi dg(t) \,.\, N_n(t) = o(1), \text{ as } n \to \infty.$$

We have, for $0 < \delta < \pi$,

$$\int_0^\pi dg(t) \,.\, N_n(t) = \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi \right) dg(t) \,.\, N_n(t)$$

$$= Q_1 + Q_2 + Q_3, \text{ say.}$$

Now

$$Q_1 = O(\int_0^{1/n} | dg(t) | \,.\, | N_n(t) |)$$

$$= O(n \int_0^{1/n} | dg(t) |),$$

$$= o\left(\frac{n p_n^\alpha}{P_n^{(\alpha)}}\right), \text{ by Lemma 2}$$

$$= o(1), \text{ as } n \to \infty.$$

$$Q_2 = O(\int_{1/n}^{\delta} | dg(t) | \,.\, | N_n(t) |)$$

$$= O\left(\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} | dg(t) | \,.\, \frac{P_\tau^{(\alpha)}}{t}\right), \text{ by Lemma 1}$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} G(t) \frac{P_\tau^{(\alpha)}}{t}\right]_{1/n}^{\delta}$$

$$+ O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} G(t) \frac{P_\tau^{(\alpha)}}{t^2} dt\right]$$

$$+ O\left[\frac{1}{P_n^{(\alpha)}} \int_{1/n}^{\delta} G(t) \frac{1}{t} dP_\tau^{(\alpha)}\right]$$

$$= Q_{2\cdot 1} + Q_{2\cdot 2} + Q_{2\cdot 3}, \text{ say.}$$

$$Q_{2\cdot 1} = O\left[\frac{1}{P_n^{(\alpha)}} G(\delta) \frac{P_{(1/\delta)}^{(\alpha)}}{\delta}\right] + O\left[\frac{1}{P_n^{(\alpha)}} G\left(\frac{1}{n}\right) \frac{P_n^{(\alpha)}}{\frac{1}{n}}\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}} \cdot \frac{p_{(1/\delta)}^{\alpha}}{\delta}\right] + o\left[\frac{1}{P_n^{(\alpha)}} \cdot \frac{p_n^{\alpha}}{\frac{1}{n}}\right]$$

$$= o(1) + o(1) = o(1), \text{ as } n \to \infty.$$

$$Q_{2\cdot2} = O\left[\frac{1}{P_n^{(\alpha)}}\int_{1/n}^{\delta} G(t)\,.\,\frac{P_\tau^{(\alpha)}}{t^2}\,dt\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}}\int_{1/n}^{\delta} \frac{p_\tau^{\alpha}}{t^2}\,dt\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}}\int_{1/\delta}^{n} p_u^{\alpha}\,du\right]$$

$$= o\left[\frac{1}{P_n^{(\alpha)}}\sum_{k=0}^{n} p_k^{\alpha}\right]$$

$$= o(1), \text{ as } n \to \infty.$$

$$Q_{2\cdot3} = o\left[\frac{1}{P_n^{(\alpha)}}\int_{1/n}^{\delta} \frac{p_\tau^{\alpha}}{P_\tau^{(\alpha)}}\,.\,\frac{1}{t}\,dP_\tau^{(\alpha)}\right]$$

$$= o(1), \text{ as in } I_{2\cdot3}.$$

Thus, $$Q_2 = o(1), \text{ as } n \to \infty.$$

Finally, since the method of summability is regular, we have $Q_3 = o(1)$, as $n \to \infty$, by Riemann-Lebesgue theorem.

This proves Theorem 2.

PROOF OF THEOREM 3 — Here also, we have to show, under the hypothesis, that

$$\int_0^{\pi} \psi(t)\,.\,\bar{N}_n(t)\,dt = o(1), \text{ as } n \to \infty.$$

We set

$$\int_0^{\pi} \psi(t)\,.\,\bar{N}_n(t)\,dt = \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi}\right]\psi(t)\,.\,\bar{N}_n(t)\,dt$$

$$= M_1 + M_2 + M_3, \text{ say.}$$

$$M_1 = \int_0^{1/n} \psi(t) \,.\, \bar{N}_n(t)\, dt$$

$$= \int_0^{1/n} \frac{\psi(t)}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \frac{\cos (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t}\, dt$$

$$= - \frac{1}{2\pi P_n^{(\alpha)}} \int_0^{1/n} \psi(t) \sum_{k=0}^{n} p_k^\alpha \frac{\cos \frac{1}{2}t - \cos (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t}\, dt$$

$$+ \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \int_0^{1/n} \psi(t) \cot \tfrac{1}{2}t\, dt.$$

Now

$$\frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \frac{\cos \frac{1}{2}t - \cos (n - k + \frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$= \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \sum_{\nu=0}^{n-k} 2 \sin \nu t$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha \sum_{\nu=0}^{n} |\sin \nu t|\right]$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^\alpha (n - k)\right]$$

$$= O(n), \text{ for } 0 \leqslant t \leqslant \pi.$$

Also, since the conjugate function exists,

$$\frac{1}{2\pi} \int_0^{1/n} \psi(t) \cot \tfrac{1}{2}t\, dt = o(1).$$

Therefore,

$$M_1 = O[n \int_0^{1/n} |\psi(t)|\, dt] + o(1)$$

(*equation continued on p. 59*)

$$= o\left[\frac{n p_n^{\alpha}}{P_n^{(\alpha)}}\right] + o(1)$$

$$= o(1), \text{ as } n \to \infty.$$

Now, for $\frac{1}{n} \leqslant t \leqslant \delta$,

$$M_2 = O[\int_{1/n}^{\delta} |\psi(t)| \cdot |\bar{N}_n(t)\, dt]$$

$$= O\left[\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{P_\tau^{(\alpha)}}{t P_n^{(\alpha)}}\right]$$

$$= o(1), \text{ as in } I_2.$$

Also, $M_3 = o(1)$, as $n \to \infty$, by Riemann-Lebesgue theorem and the regularity of the method of summation.

This proves Theorem 3.

Remarks : It is evident that for $\alpha = 1$ and $p_n = \frac{1}{n+1}$ Theorem 2 reduces to the following theorem of Tripathi (1963).

Theorem B — The derived series of the Fourier series of the function $f(x)$ is summable by harmonic means to the sum $f'(x)$ at a point x at which

$$G(t) = o\left(\frac{t}{\log \frac{1}{t}}\right), \quad t \to +0,$$

where $f'(x)$ denotes the first generalized differential coefficient of $f(t)$ at $t = x$.

Again, for $\alpha = 1$ and $p_n = \frac{1}{n+1}$, Theorem 1 and Theorem 3 reduce to the classical results for harmonic summability of Siddiqi (1948) for the Fourier series and its conjugate series.

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