INTERNAL INSTABILITY IN SECOND ORDER ELASTICITY

by SUKHENDU DEY *and* SUSHIL KUMAR ADDY, *Department of Physics and Mathematics, Indian School of Mines, Dhanbad*

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Considering the second order elasticity, the internal instability of an incompressible homogeneous elastic medium under initial stress is discussed. The medium is supposed to possess finite isotropy, but incremental orthotropy. It is shown that due to finite isotropy internal instability of the first kind is not possible. But internal instability of the second kind occurs and it begins to appear when a particular relation between the hydrostatic part and the difference of the deviatoric parts of the initial stress system holds. It is also shown that internal instability can be brought about by an initial hydrostatic stress only when its magnitude attains a value twice the rigidity of the material. It is known that in the case of first order elasticity, the hydrostatic stress does not play any role in distortion and consequently it cannot be responsible for internal instability. But when we consider second order elasticity, the hydrostatic stress shows a significant effect on the occurrence of internal instability. This is due to the fact that the strain energy developed by the hydrostatic stress is a function of second order strain components which are neglected in the linearized theory.

INTRODUCTION

Internal instability is a physical phenomenon associated with initial stress. It may occur in a medium of infinite extent or in a finite region confined between rigid boundaries. From the mathematical viewpoint it is due to the hyperbolic or mixed hyperbolic-elliptic nature of the equations for the incremental field beyond certain critical values of the initial stress. Biot (1940) has shown the existence of such internal instability in the theory of acoustic propagation under initial stress. He explained the existence and nature of this phenomenon in a later detailed discussion (Biot 1963). There are two physically distinct cases : internal instability of the first kind, and internal instability of the second kind. Biot (1965, pp. 192-204) has discussed both these cases, considering the first order elasticity.

In this paper, we have discussed the internal instability of an elastic medium taking the second order elasticity into consideration. The medium is incompressible and homogeneous and it is isotropic for finite deformations, but orthotropic for incremental ones. We have shown that due to finite isotropy, internal instability of the first kind is not possible. But internal instability of the second kind occurs and it begins to appear when the relation

$$P = \sqrt{4H(2\mu - H)}$$

holds, where H is the hydrostatic part and P is the difference of the deviatoric parts of the initial stress system, μ being the regidity of the material. We have also shown that an initial hydrostatic stress of magnitude 2μ alone can initiate internal instability of the second kind in a medium possessing the properties stated above.

The paper has thus brought an important fact to light, viz. the role of hydrostatic stress in internal instability.

Formulation and Solution of the Problem

Let us consider a homogeneous and incompressible elastic medium of finite or infinite extent. The medium is supposed to possess finite isotropy, but incremental orthotropy. The coordinate axes are oriented along the directions of elastic symmetry. The initial stress components in the medium are the principal compressive stresses P_{11} and P_{22} along the x and y axes respectively.

In plain strain, incremental stress components s_{ij} satisfy the equilibrium equations (Biot 1965, p. 38)

$$\left.\begin{aligned} \frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} &= 0 \\ \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

where $P = P_{11} - P_{22}$

and $\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

is the rotation, u, v being the displacement components along the x and y axes respectively.

For the orthotropic medium, incremental stress components s_{ij} and incremental strain components e_{ij} are connected by the relations (Biot 1965, p. 101)

$$\left.\begin{aligned} s_{11} &= B_{11}e_{xx} + B_{12}e_{yy} \\ s_{22} &= B_{21}e_{xx} + B_{22}e_{yy} \\ s_{12} &= 2Qe_{xy} \end{aligned}\right\} \qquad \ldots(2)$$

where $e_{xx} = \frac{\partial u}{\partial x}$

$e_{yy} = \frac{\partial v}{\partial y}$

and $e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$.

The incremental elastic coefficients B_{ij} can be expressed in terms of the basic coefficients λ, μ, D and F as follows (Biot 1965, p. 111) :

$$\left.\begin{aligned} B_{11} &= (2\mu + \lambda)(1 + \epsilon_{11} - \epsilon_{22}) + 2D\epsilon_{11} + 2F\epsilon_{22} \\ B_{22} &= (2\mu + \lambda)(1 + \epsilon_{22} - \epsilon_{11}) + 2D\epsilon_{22} + 2F\epsilon_{11} \\ B_{12} - P_{11} &= B_{21} - P_{22} = \lambda + 2F(\epsilon_{11} + \epsilon_{22}) \end{aligned}\right\} \quad \ldots(3)$$

where ϵ_{11} and ϵ_{22} are the second order incremental strain components along the x and y axes respectively and are given by (Boit 1965, p. 12)

$$\epsilon_{11} = e_{xx} + e_{xy}\omega + \tfrac{1}{2}\omega^2$$

$$\epsilon_{22} = e_{yy} - e_{xy}\omega + \tfrac{1}{2}\omega^2.$$

The non-linear stress-strain relations in the incompressible medium under initial stress are not available. We shall derive these equations in the limiting case. For this, we consider the condition of incompressibility

$$e_{xx} + e_{yy} = 0 \quad \ldots(4)$$

with λ tending to infinity in such a way that

$$\lim_{\lambda \to \infty} \lambda(e_{xx} + e_{yy}) = s_0$$

where s_0 is a finite quantity.

Substituting eqns. (3) into eqns. (2), considering the condition of incompressibility (4) and retaining terms only up to second order, we get the following stress-strain relations :

$$\left.\begin{aligned} \tfrac{1}{2}(s_{11} - s_{22}) &= s_{11} - s = (2\mu - H)\, e_{xx} + s_0(e_{xx} - e_{yy}) \\ \tfrac{1}{2}(s_{22} - s_{11}) &= s_{22} - s = (2\mu - H)\, e_{yy} - s_0(e_{xx} - e_{yy}) \\ s_{12} &= 2\mu e_{xy} \end{aligned}\right\} \quad \ldots(5)$$

where $s = \tfrac{1}{2}(s_{11} + s_{22})$ and $H = \tfrac{1}{2}(P_{11} + P_{22})$, which are hydrostatic parts of the incremental and initial stress system respectively.

Here, we have taken the linear relation between the shear stress and strain making Q equal to μ.

Neglecting $s_0(e_{xx} - e_{yy})$, eqns. (5) reduce to

$$\left.\begin{aligned} s_{11} - s &= (2\mu - H)e_{xx} \\ s_{22} - s &= (2\mu - H)e_{yy} \\ s_{12} &= 2\mu e_{xy} \end{aligned}\right\} \quad \ldots(6)$$

The condition of incompressibility (4) is satisfied by putting

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial y} \\ v &= \frac{\partial\phi}{\partial x} \end{aligned}\right\} \qquad \ldots(7)$$

where Φ is a function of x and y.

With the help of eqns. (1), (6) and (7), we get the following equations :

$$\left.\begin{aligned} \frac{\partial s}{\partial x} &= \frac{\partial}{\partial y}\left[(\mu - P_{22})\frac{\partial^2\phi}{\partial x^2} + (\mu + \tfrac{1}{2}P)\frac{\partial^2\phi}{\partial y^2}\right] \\ \frac{\partial s}{\partial y} &= -\frac{\partial}{\partial x}\left[(\mu - \tfrac{1}{2}P)\frac{\partial^2\phi}{\partial x^2} + (\mu - P_{11})\frac{\partial^2\phi}{\partial y^2}\right]. \end{aligned}\right\} \qquad \ldots(8)$$

Here, P may be regarded as the difference of the deviatoric parts of the initial stress system.

Differentiating eqns. (8) with respect to y and x respectively, the term s can be eliminated, and the following equation in terms of Φ is obtained :

$$(\mu - \tfrac{1}{2}P)\frac{\partial^4\phi}{\partial x^4} + 2(\mu - H)\frac{\partial^4\phi}{\partial x^2\partial y^2} + (\mu + \tfrac{1}{2}P)\frac{\partial^4\phi}{\partial y^4} = 0. \qquad \ldots(9)$$

The occurrence of internal instability is closely connected with the existence of hyperbolic solutions of eqn. (9). We put

$$\Phi = \psi(x - \xi y) \qquad \ldots(10)$$

where ξ is a non-zero real quantity.

Substituting eqn. (10) into eqn. (9), we obtain the following characteristic equation :

$$\xi^4 + 2m\xi^2 + k^2 = 0 \qquad \ldots(11)$$

$$\left.\begin{aligned} \text{with} \quad m &= \frac{\mu - H}{\mu + \frac{1}{2}P} \\ \text{and} \quad k^2 &= \frac{\mu - \frac{1}{2}P}{\mu + \frac{1}{2}P}. \end{aligned}\right\} \qquad \ldots(12)$$

The roots ξ^2 of eqn. (11) are

$$\xi_1^2 = -m + \sqrt{m^2 - k^2}, \quad \xi_2^2 = -m - \sqrt{m^2 - k^2}. \qquad \ldots(13)$$

The existence of a real root ξ makes a solution of the hyperbolic type (10) possible. Two different cases arise.

Case I : $m > 0$, $k^2 < 0$—The root ξ_1^2 is positive and so ξ_1 is real.

Case II : $m < 0, m^2 > k^2 > 0$—Both ξ_1^2 and ξ_2^2 are positive and so both ξ_1 and ξ_2 are real.

These two cases correspond to two types of phenomena. They may be called the internal instability of the first and second kind.

Internal Instability of the First Kind

This case occurs when

$$m > 0, \; k^2 < 0. \qquad \ldots(14)$$

These conditions may be written in the equivalent form

$$\mu > H, \; P > 2\mu. \qquad \ldots(15)$$

For finite isotropy, Q is given by (Biot 1965, p. 93)

$$Q = \tfrac{1}{2}(P_{22} - P_{11}) \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \qquad \ldots(16)$$

where λ_1 and λ_2 are the extension ratios of the finite initial strain along the x and y axes respectively.

We can write eqn. (16) in the form

$$2\mu = P \frac{\lambda_2^2 + \lambda_1^2}{\lambda_2^2 - \lambda_1^2}\,. \qquad \ldots(17)$$

It is easy to see from eqn. (17) that P cannot be greater than 2μ.

Hence, the internal instability of the first kind is not possible in a medium possessing finite isotropy. This result is in agreement with that of Biot, who showed that in the first order elasticity, internal instability is not possible in a medium which is isotropic for finite deformations.

Internal Instability of the Second Kind

This case occurs when

$$m < 0, \; m^2 > k^2 > 0 \qquad \ldots(18)$$

These conditions may be written in the equivalent form

$$H > \mu, \; 2\mu > P > \sqrt{4H(2\mu - H)}\,. \qquad \ldots(19)$$

If H is kept constant, the value of P as a function of ξ is given by eqn. (11). The variation of P with respect to ξ is shown in Fig. 1. As $m < 0$, the value of P goes through a minimum P_{min}, which is given by

$$P_{min} = \sqrt{4H(2\mu - H)} \qquad \ldots(20)$$

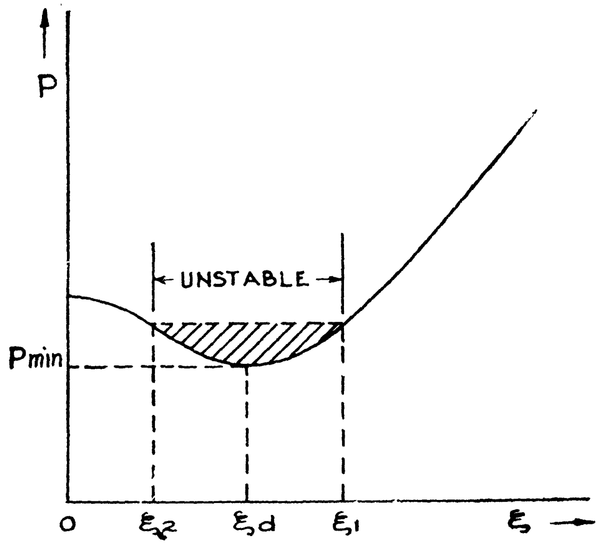

FIG. 1. Diagrammatic representation of internal instability of the second order

and the corresponding ξ_d is given by

$$\xi_d = \sqrt{\frac{H - \mu}{\mu + \frac{1}{2}\sqrt{4H(2\mu - H)}}}\,. \qquad \ldots(21)$$

If $P > P_{min}$, there is a region of instability in the range

$$\xi_2 < \xi < \xi_1$$

as shown in Fig. 1.

Incipient instability occurs for

$$P = P_{min} = \sqrt{4H(2\mu - H)}\,. \qquad \ldots(22)$$

The corresponding slopes of the two characteristic directions are

$$\xi = \pm\,\xi_d. \qquad \ldots(23)$$

At this point, the characteristic equation (11) has a double root and so incipient instability is given by the equation

$$m^2 - k^2 = 0 \qquad \ldots(24)$$

or

$$P = \sqrt{4H(2\mu - H)}. \qquad \ldots(25)$$

If H is a function of P, condition (25) for incipient instability remains valid, but it is now an intrinsic equation for P.

When $P = 0$, i.e., when the initial stress system is solely hydrostatic, incipient instability is given by

$$H = 2\mu. \qquad \ldots(26)$$

Hence, if we consider the second order elasticity, we see that under an hydrostatic stress 2μ, internal instability begins to show its occurrence in a medium which is isotropic for finite deformations, but orthotropic for incremental ones.

References

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