VIBRATIONS OF A TWISTED STRIP UNDER CENTRIFUGAL AND UNSTEADY AERODYNAMIC FORCE FIELD

by B. B. SHARMA, *D.A.V. College, Muzaffarnagar (U.P.)*

(*Received* 22 *January* 1977)

In this paper, differential equations have been derived for the coupled torsional and longitudinal vibrations of a pretwisted slender beam under unsteady aerodynamic couplings in a steady flow of an incompressible flow in a centrifugal force field. A method based on the series solution is used to obtain the critical speed of the flow and the corresponding fundamental frequency of vibrations. It has been found that the fundamental frequency of oscillation may increase considerably in the case of unsteady aerodynamic coupling.

NOTATIONS

l = length

S = area of cross-section of the wing

G = shear modulus

J = a constant depending upon the cross-section of the bar

β = the twist in radian per unit length of x-axis

E = Young's modulus of elasticity

L = aerodynamic lift about the elastic axis

N = aerodynamic moment per unit span about the elastic axis

ρ = density of the material

t = time.

INTRODUCTION

A pretwisted slender beam that could represent a turbine blade of simple geometry is considered. The beam is attached to a disc of radius r_0 and the disc rotates with a constant angular velocity Ω. The coupling between longitudinal and torsional motions is analysed for a pretwisted bar using a beam type theory. It is known that when a thin bar is under torsion, there is a slight decrease in distance between cross-sections. Thus, a normal stress is produced in each longitudinal fibre, which is not parallel to the axis of the bar and hence there is a stress component which produces an additional torque.

DIFFERENTIAL EQUATIONS

The governing differential equations for the coupled torsional and longitudinal vibrations of a pretwisted slender beam are (Sharma 1975) :

$$\left.\begin{aligned} & ES\frac{\partial^2 h}{\partial x^2} + EI_0\beta\frac{\partial^2 \alpha}{\partial x^2} - \rho S\frac{\partial^2 h}{\partial t^2} = 0 \\ & (GJ + EI_1\beta^2)\frac{\partial^2 \alpha}{\partial x^2} + EI_0\beta\frac{\partial^2 h}{\partial x^2} - \rho I_0\frac{\partial^2 \alpha}{\partial t^2} = 0 \end{aligned}\right\} \qquad \ldots(1)$$

$$I_0 = \int_S r^2\, ds, \quad I_1 = \int_S r^4\, ds$$

where r_0 is the distance of the longitudinal fibre from the axis of the bar.

When the elastic force is included, the blade will be deformed. Also, in a flow of speed U, the blade will have some deformations due to the unsteady aerodynamic forces. After taking into consideration the effect of centrifugal force, eqns. (1) become

$$\left.\begin{aligned} & ES\frac{\partial^2 h}{\partial x^2} + EI_0\beta\frac{\partial^2 \alpha}{\partial x^2} - \rho S\frac{\partial^2 h}{\partial t^2} + \rho S\Omega^2\{r_0(l - x) \\ & \quad + \tfrac{1}{2}(l^2 - x^2)\}\frac{\partial h}{\partial x} + L = 0 \\ & (GJ + EI_1\beta^2)\frac{\partial^2 d}{\partial x^2} + EI_0\beta\frac{\partial^2 h}{\partial x^2} - \rho I_0\frac{\partial^2 \alpha}{\partial t^2} + N = 0 \end{aligned}\right\} \qquad \ldots(2)$$

where L and N are given by

$$L = L_1 + L_2 + L_3$$

$$N = (\tfrac{1}{2} + a_h)\, bL_1 + a_h bL_2 - (\tfrac{1}{2} - a_h)\, bL_3 + M_a$$

$$L_1 = 2\pi b\rho U^2\alpha\phi(\Gamma)$$

$$\phi(\Gamma) = 0 \text{ if } \Gamma < 0.$$

A lift, L_2, with centre of pressure at the mid-chord, of amount equal to the apparent mass $\rho\pi b^2$ times the vertical acceleration at the mid-chord point is given by

$$L_2 = \rho\pi b^2\frac{\partial^2}{\partial t^2}(h - a_h b\alpha).$$

A lift force centre of pressure at the $\frac{3}{4}$ chord point, of the nature of centrifugal force of amount equal to the apparent mass times $U\frac{\partial \alpha}{\partial t}$, is

$$L_3 = \rho\pi b^2 U\frac{\partial \alpha}{\partial t}.$$

A nose-down couple equal to the apparent moment of inertia times $\rho\pi b^4/8$, the angular acceleration $\dfrac{\partial^2\alpha}{\partial t^2}$, is

$$M_a = -\frac{\rho\pi b^4}{8} \cdot \frac{\partial^2\alpha}{\partial t^2}.$$

Here, the airfoil has two degrees of freedom, a vertical translation h, called bending positive downward, and a rotation α, called pitching, positive nose up, about an axis located at a distance $a_h b$ from the mid-chord point, a_h being positive towards the trailing edge.

Determination of Natural Frequencies and Critical Speed

Equations (2) are now put in terms of dimensionless variables by introducing $\xi = \dfrac{x}{l}$ with the following substitutions

$$\alpha_0 = \frac{E}{\rho l^2};\quad \nu = \Omega^2,\ \delta = \frac{EI_0\beta}{\rho S l^2}$$

$$m = \frac{GJ + EI_1\beta^2}{\rho I_0 l^2},\quad n = \frac{E\beta}{\rho l^2}$$

$$K_1 = \frac{\pi b^2}{S},\ K_2 = \frac{2\pi b}{S}\,\phi(\Gamma),\quad K_3 = \frac{\pi b^3 a_h}{I_\theta}$$

$$K_4 = (\tfrac{1}{2} + a_h)\, 2\pi b^2\, \frac{\phi(\Gamma)}{I_0}$$

$$K_5 = \frac{a_h^2\, b^4\pi}{I_0},\quad K_6 = (\tfrac{1}{2} - a_h)\frac{\pi b^3}{I_0},\quad K_7 = \frac{\pi b^4}{8I_0}$$

$$\left.\begin{aligned} &\alpha_0 \frac{\partial^2 h}{\partial\xi^2} + \nu\left\{\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\,\xi - \tfrac{1}{2}\,\xi^2\right\}\frac{\partial h}{\partial\xi} - \frac{\partial^2 h}{\partial t^2} + \delta\,\frac{\partial^2\alpha}{\partial\xi^2} \\ &\quad + K_2U^2\alpha + K_1\left(\frac{\partial^2 h}{\partial t^2} - a_h b\,\frac{\partial^2\alpha}{\partial t^2}\right) + K_1 U\,\frac{\partial\alpha}{\partial t} = 0 \\ &n\,\frac{\partial^2 h}{\partial\xi^2} + m\,\frac{\partial^2\alpha}{\partial\xi^2} - \frac{\partial^2\alpha}{\partial t^2} + K_4U^2\alpha + K_3\,\frac{\partial^2 h}{\partial t^2} - K_5\,\frac{\partial^2\alpha}{\partial t^2} \\ &\quad - K_6 U\,\frac{\partial\alpha}{\partial t} - K_7\,\frac{\partial^2\alpha}{\partial t^2} = 0. \end{aligned}\right\} \qquad \text{...(3)}$$

The solutions of eqns. (3) are assumed to be of the form

$$\left.\begin{aligned} h(\xi, t) &= Af(\xi)\, e^{i\omega t} \\ \alpha(\xi, t) &= B\phi(\xi)\, e^{i\omega t} \end{aligned}\right\} \qquad \text{...(4)}$$

where A and B are constants, which are not independent, and $f(\xi)$ and $\phi(\xi)$ are functions of ξ only.

The functions $f(\xi)$ and $\phi(\xi)$ are to satisfy the boundary conditions, which are

$$\left.\begin{aligned} h = \alpha = 0 \quad & \text{at } \xi = 0 \\ \frac{\partial h}{\partial \xi} = \frac{\partial \alpha}{\partial \xi} = 0 \quad & \text{at } \xi = 1. \end{aligned}\right\} \qquad \ldots(5)$$

For an approximate determination of the fundamental frequency, $f(\xi)$ is chosen as the shape function for the fundamental mode of uncoupled longitudinal vibrations and $\phi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration of a uniform cantilever beam. The following shape functions satisfy the boundary conditions in (5) :

$$f(\xi) = 2\xi - 3\xi^2 + \tfrac{4}{3}\xi^3$$

$$\phi(\xi) = \sin\left(\frac{\pi\xi}{2}\right).$$

Substituting eqns. (4) in (3), we obtain

$$\left.\begin{aligned} & A\left[\alpha_0 \frac{d^2 f}{d\xi^2} + \nu\left\{\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\,\xi - \tfrac{1}{2}\,\xi^2\right\}\frac{df}{d\xi} + \omega^2 f - K_1 f\omega^2\right] \\ & \qquad + B\left[\delta\frac{d^2\phi}{d\xi^2} + K_2 U^2\phi + K_1 a_h b\omega^2\phi + K_1 iU\omega\phi\right] = 0 \\ & A\left[n\frac{d^2 f}{d\xi^2} - K_3\omega^2 f\right] + B\left[m\frac{d^2\phi}{d\xi^2} + \omega^2\phi + K_4 U^2\phi + K_5\omega^2\phi \right. \\ & \qquad \left. - K_6 iU\omega\phi + K_7\omega^2\phi\right] = 0. \end{aligned}\right\} \qquad \ldots(6)$$

Eqns. (6) can be solved for ω^2, but the result is a function of ξ, as f and ϕ are not the exact shape functions.

This difficulty can be overcome by multiplying the first and second of eqns. (6) by f and ϕ respectively and integrating the result with respect to ξ from 0 to 1. Thus, we obtain the following equations :

$$\left.\begin{aligned} & A\,[-\,a_1 + a_2 + a_3\omega^2 - a_4\omega^2] + B\,[-\,a_5 + a_6 U^2 + a_7\omega^2 \\ & \qquad + a_8 iU\omega] = 0 \\ & A\,[-\,a_9 - a_{10}\omega^2] + B\,[-\,a_{11} + a_{12}\omega^2 + a_{13}U^2 + a_{14}\omega^2 \\ & \qquad - a_{15} iU\omega + a_{16}\omega^2] = 0 \end{aligned}\right\} \qquad \ldots(7)$$

where

$$a_1 = -\,\alpha_0\int_0^1 \frac{d^2 f}{d\xi^2}\cdot f\,d\xi = \alpha_0\int_0^1 \left(\frac{df}{d\xi}\right)^2 d\xi$$

$$a_2 = \nu \int_0^1 \left(\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\,\xi - \tfrac{1}{2}\xi^2\right)\frac{df}{d\xi}\, f\, d\xi$$

$$a_3 = \int_0^1 f^2\, d\xi, \quad a_4 = K_1 \int_0^1 f^2\, d\xi$$

$$a_5 = -\,\delta \int_0^1 \frac{d^2\phi}{d\xi^2}\, f d\xi = \delta \int_0^1 \frac{d\phi}{d\xi}\,\frac{df}{d\xi}\, d\xi$$

$$\alpha_6 = K_2 \int_0^1 f\phi\, d\xi, \quad a_7 = K_1 a_h b \int_0^1 f\phi\, d\xi$$

$$a_8 = K_1 \int_0^1 f\phi\, d\xi, \quad a_{10} = K_3 \int_0^1 f\phi\, d\xi$$

$$a_9 = -\, n \int_0^1 \frac{d^2 f}{d\xi^2}\,\phi\, d\xi = n \int_0^1 \frac{df}{d\xi}\,\frac{d\phi}{d\xi}\, d\xi$$

$$a_{11} = -\, m \int_0^1 \frac{d^2\phi}{d\xi^2}\,\phi\, d\xi = m \int_0^1 \left(\frac{d\phi}{d\xi}\right)^2 d\xi$$

$$a_{12} = \int_0^1 \phi^2\, d\xi; \quad a_{13} = K_4 \int_0^1 \phi^2\, d\xi$$

$$a_{14} = K_5 \int_0^1 \phi^2\, d\xi$$

$$a_{15} = -\, K_6 \int_0^1 \phi^2\, d\xi$$

$$a_{16} = K_7 \int_0^1 \phi^2\, d\xi$$

For a nontrivial solution, both A and B must not vanish. Consequently, the determinant of the coefficients of eqn. (7) must be zero. This determinant being complex, both real and imaginary parts must vanish. On setting the determinant equal to zero,

$$\begin{vmatrix} -\, a_1 + a_2 + (a_3 - a_4)\,\omega^2 & -\, a_5 + a_6 U^2 + a_7\omega^2 + a_8 U\omega i \\ -\, a_9 - a_{10}\omega^2 & \begin{array}{l} -\, a_{11} + a_{12}\omega^2 + a_{13}U^2 + a_{14}\omega^2 \\ -\, a_{15}U\omega i + a_{16}\omega^2 \end{array} \end{vmatrix} = 0$$

i.e.

$$\left.\begin{aligned} A_1\omega^4 - (C_1 + C_2U^2)\,\omega^2 + (E_1 + E_2U^2) &= 0 \\ - B_1\omega^2 + B_2 &= 0 \end{aligned}\right\} \qquad \ldots(8)$$

$$A_1 = (a_3 - a_4)\,(a_{12} + a_{14} + a_{16}) + a_7a_{10}$$

$$C_1 = a_{11}(a_3 - a_4) + (a_1 - a_2)\,(a_{12} + a_{14} + a_{16}) + a_5a_{10} + a_7a_9$$

$$C_2 = a_{13}(a_4 - a_3) - a_6a_{10}$$

$$E_1 = a_{11}(a_1 - a_2) - a_5a_9$$

$$E_2 = a_{13}(a_2 - a_1) + a_6a_9$$

$$B_1 = a_{15}(a_3 - a_4) - a_8a_{10}$$

$$B_2 = a_{15}(a_1 - a_2) + a_8a_9.$$

The second of eqns. (8) gives

$$\omega^2 = \frac{B_2}{B_1} \qquad \ldots(9)$$

Substituting this value into the first of eqns. (8),

$$U^2 = \frac{\left[\left(\frac{B_2}{B_1}\right)C_1 - \left(\frac{B_2}{B_1}\right)^2 A_1 - E_1\right]}{\left[E_2 - \left(\frac{B_2}{B_1}\right)C_2\right]}.$$

The smaller of the two values of ω given by eqn. (9) is an upper bound for the frequency of the fundamental mode of vibration and the larger of these values is an upper bound for the next higher mode of vibrations.

Substituting the value of ω^2 in the first of eqns. (8), the value of the critical speed will be obtained.

Thus, it has been shown that the method based on series solution is suitable for obtaining the critical speed and frequency corresponding to the fundamental mode of vibration for the inertially coupled or uncoupled torsional and longitudinal vibrations of twisted strip under unsteady aerodynamic forces.

Numerical Example

A numerical example for the coupled torsional and longitudinal vibrations of a pretwisted rotating beam under unsteady aerodynamic couplings is now presented. The frequencies and speeds are computed from eqns. (9) and (8) respectively. The cross-section of the beam is taken as a rectangle having width $2b$ and depth $2h$. The formula $J = 4K_1Sh^2$ is used to calculate J, where $K_1 = 0.312$ for $(h/b) = 0.1$ and 1/3 for $(h/b) = 1/15$ and $1/20$ (Timoshenko and Goodier 1951). The other physical constants of the blade are given below :

$$E = 30 \times 10^6 \text{ lb/in}^2, \quad G = 11.25 \times 10^6 \text{ lb/in}^2$$

$$r_0 = 32 \text{ in}, \quad \Omega = 314 \text{ sec}^{-1}$$

$$l = 8 \text{ in}, \quad b' = 1 \text{ in}$$

$$a_0 = b = 0.23885 \text{ in}, \quad \rho = 0.00082 \text{ lb/in}^3$$

$$\phi(\Gamma) = 1 - \frac{2}{4 + \Gamma}, \quad \Gamma = 2$$

$$a_h = -\ 0.015$$

With these values, the various a and A values are computed from eqn. (9). The value of frequency is

$$\omega^2 = 6.0635 \times 10^9$$

Now substituting this value of ω^2 in (8), the value of the critical speed is obtained as

$$U^2 = 1.74293 \times 10^{12}.$$

References

Sharma, B. B. (1976). Coupled torsional and longitudinal vibrations of a pretwisted beam in a centrifugal force field. *Indian J. pure appl. Math.*, 7.

Timoshenko, and Goodier, J. N. (1951). Theory of Elasticity. McGraw-Hill Book Co.. Inc., New York.

Fung, F. C. (1955). An Introduction to the Theory of Aeroelasticity. John Wiley and Sons, Inc., New York.