A PROBLEM OF A DETERMINISTIC QUEUE

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(*Received* 22 *January* 1977)

Ships are arriving at different points at a seaport to seek services. It has been assumed that the arrival and service times of customers are fixed and further that the times in going from different points to service counters are different. The problem is to find (i) the number of ships waiting in the queue, and (ii) the total waiting time. A probability model has been formulated and the above problem has been solved for this model.

FORMULATION

We take as fixed the arrival time and the service time of customers (Saaty 1961). Customers are assumed to take different times to reach the service point. For example, there are many ships full of oil at different points of the port. Their arrival rate at a particular point of the port is fixed; then they are stationed at different points in the port. After this, the ships are brought to a particular point for the service (emptying). When a particular ship is removed from the service point, a signal is given to the next ship to reach the service point.

Let the ships arrive in a single channel at regular intervals of time a (i.e., arrival rate $1/a$) and are served at regular time intervals of length b (service rate $1/b$)

If $\quad b + k_n < a$, i.e., $\dfrac{b + k_n}{a} < 1$, no ship will have to wait.

If $\quad \dfrac{b + k_n}{a} > 1$, the number of waiting customers will increase indefinitely.

We suppose that $\dfrac{b + k_n}{a} < 1$ and the operation begins with a line of i ships. We have $i \geqslant 2$, for if $i = 1$, that ship will finish service before the last ship arrives. All i ships will be served by the end of a time interval

$$ib + \sum_{n=1}^{i} k_n. \qquad \ldots(1)$$

By then $\dfrac{1}{a}\{ib + \sum_{n=1}^{i} k_n\}$ ships would have arrived and would have to wait. If this expression is a fraction, it should be taken to be the largest integer which is less than

$$\frac{1}{a}\left\{ib + \sum_{n=1}^{i} k_n\right\} = n_1 \text{ (say)}$$

since ships do not arrive in fractions, the last arriving ship waits for the service facility to complete its service.

The service times of these ships which will first wait in the line is

$$n_1 b + \sum_{j=i+1}^{n_1} k_j = \frac{b}{a}\left[ib + \sum_{n=1}^{i} k_n\right] + \sum_{j=i+1}^{\frac{1}{a}[ib+\sum_{n=1}^{i} k_n]} k_j \qquad \text{...(2a')}$$

and again during this time

$$\frac{1}{a}\left\{\frac{b}{a}\left[ib + \sum_{n=1}^{i} k_n\right] + \sum_{j=i+1}^{\frac{1}{a}[ib+\sum_{n=1}^{i} k_n]} k_j\right\} \qquad \text{...(2b')}$$

ships will have arrived.

The length of the line of ships will become zero after a lapse of time in which every ship which has waited has also been served.

Let A new arrivals have to wait in line. Then certainly the $(A + 1)$th arrival occurs after the service time of $i + A$ customers, i.e.

$$(i + A)b + \sum_{n=1}^{i+A} k_n \leqslant a(A + 1).$$

This gives

$$A = \frac{ib - a}{a - b} + \left(\frac{1}{a - b}\sum_{n=1}^{i+A} k_n\right) + 1. \qquad \text{...(3)}$$

Therefore, the length of time required to serve all items which have waited is

$$b\left[i + \frac{ib - a}{a - b} + \left(\frac{1}{a - b}\sum_{n=1}^{i+A} k_n\right) + 1\right] \qquad \text{...(4a')}$$

$$= b\left[\frac{ia - b}{a - b} + \frac{1}{a - b}\sum_{n=1}^{i+A} k_n\right]. \qquad \text{...(4b')}$$

Again, this includes the first of the i initial ships

We start the operation by giving signal to the first ship to arrive at the service point from the initially waiting ships, if it become empty and no ship arrives during this time (no ship is ready at the port for the purpose). On the arrival of another ship, signal is given to it to reach the service point. If a ship arrives during the service of this ship or that of a subsequent ship, the arriving ship will be taken into service in the manner mentioned above. During the time of this service, it is possible that a ship joins the system, depending on the magnitude of $a - (b + k_n)$. The arriving ship will enter service once it becomes vacant, and all ships joining the system are to be emptied. When no ships are waiting in the system, the system is idle till there are i ships in the system.

The system is again in action till all the ships become empty; then again the system is idle when there is no ship in the system.

Consider the increments of service time gained by the fact that $a > b + k_n$. Thus, $a - (b + k_n)$ time units are saved. The time saved up to the turn of nth ship (before it is given signal to reach the service point) is

$$(n - 1)(a - b) - \sum_{n=1}^{n-1} k_n$$

and the time required to serve the i ships in the queue is

$$ib + \sum_{n=1}^{i} k_n. \qquad \ldots(5)$$

If $(n_i - 1)$ ships provide sufficient slack time, so that i ships may be served in it, then

$$(n_i - 1)(a - b) - \sum_{n=1}^{n-1} k_n = ib + \sum_{n=1}^{i} k_n. \qquad \ldots(6)$$

It gives

$$n_i = 1 + \frac{ib}{a - b} + \frac{1}{a - b}\left\{\sum_{n=1}^{i} k_n + \sum_{n=1}^{n-1} k_n\right\}. \qquad \ldots(7)$$

Thus, the total time until the service facility first becomes idle is

$$T = b_{n_i} + \sum_{n=1}^{n_i} k_n. \qquad \ldots(8)$$

If a ship reaches the arrival point at time $t < T$, then t/a ships would have crossed the arrival point before it, bringing the total to $\frac{t}{a} + i - 1$ and if k_m is the mean time taken by the ships to reach the service point, then $\frac{t}{b + k_m}$ ships would have been served during time t.

Thus, at time t in the system, there would be

$$\frac{t}{a} + (i-1) - \frac{t}{b+k_m} = t\left(\frac{1}{a} - \frac{1}{b+k_m}\right) + (i-1)$$

ships in the queue. The time spent in the line by an arriving ship is

$$W(t) = \begin{cases} 0 & \text{if } T - a \leqslant t \\ t\left[\left\{\dfrac{b+k_m-a}{a(b+k_m)}\right\} + i\right]b - a & \text{if } 0 < t \leqslant T - a \\ (k-1)(b+k_m) & \text{if } t = 0 \end{cases} \quad \text{...(9)}$$

where $\underset{t=0}{W(t)}$, i.e., $W(0)$ is the time spent in the system by the kth ship of the initial group of i ships.

Thus, the total time spent in the system by a ship, including the time spent in service, is

$$W(t) + b + k_m. \qquad \text{...(10)}$$

Probability Models

Now we consider the problem in probability terms. Suppose the reaching times of the ships at the service point are discrete and their lengths are $k_1, k_2, \ldots$ time units and the corresponding probabilities of their occurrence are $p_1, p_2, \ldots$ respectively. Then if p_n is the probability that the ship will require service of duration no longer than n, we have

$$P_n = p_1 + p_2 + \ldots + p_n$$

and hence $p_n = P_n - P_{n-1}$.

Now the expected time taken by the first i ships to reach the service point is

$$\sum_{n=1}^{i} p_n k_n = p' \quad \text{(say).} \qquad \text{...(11)}$$

Then (1) gives that all i ships will be served by the end of a time interval of length

$$ib + p'.$$

By then $\frac{1}{a}\{ib + p'\}$ ships would have arrived and would have to wait; say this number is N_1.

The service times of these ships which will first wait will now be given by

$$N_1 b + p_1' \qquad \text{...(12)}$$

where $p_1' = p_{i+1}\, k_{i+1} + p_{i+2}\, k_{i+2} + \ldots + p_{N_1}\, k_{N_1}$ ...(13)

Again, during this time, the number of arrivals is

$$\frac{1}{a}\,(N_1 b + p_1'). \qquad \ldots(14)$$

The length of the line of ships will become zero after a lapse of time in which every ship which has waited has also been served. Let A new arrivals have to wait in line, then certainly $(A + 1)$th arrival occurs after the service time of $i + A$ customers, i.e.

$$(i + A)\, b + p_2' \leqslant a(A + 1) \text{ giving } A = \frac{ib - a}{a - b} + \frac{p_2'}{a - b} + 1. \qquad \ldots(15)$$

Thus, the length of time required to serve all items which have waited is approximately given by

$$b\left\{\frac{ia - b}{a - b} + p_2'\right\} \qquad \ldots(16)$$

including the first of the i initial customers

Suppose k_m' is the maximum value of the time taken by a ship to reach the service point, such that $a > b + k_m'$. Thus, at least $a - (b + k_m')$ time is gained by the fact that $a > b + k_m'$ for every ship. Thus, the time saved up to the turn of the nth ship is at least

$$(n - 1)\,\{a - (b + k_m')\} \qquad \ldots(17)$$

{before the nth ship is given signal to reach the service point} the time required to serve i ships in the queue is

$$ib + p'.$$

If $(\bar{n}_i - 1)$ ships provide sufficient slack time, so that i ships may be served in it, then

$$(\bar{n}_i - 1)\,\{a - (b + k_m')\} = ib + p'$$

gives

$$\bar{n}_i = \frac{a + b(i - 1) + k_m' + p'}{a - b + k_m'} \qquad \ldots(18)$$

Thus, the service facility will be idle at the most after time

$$T = b\bar{n}_i = b\left\{\frac{a + b(i - 1) + k_m' + p'}{a - b + k_m'}\right\}. \qquad \ldots(19)$$

In this case also, the waiting time of a ship is given by eqn. (9) and the time spent by a ship is given by eqn. (10).

CASE OF c SERVICE CHANNELS

Now consider a case of c service channels. The time spent in the service of the first i ships is

$$\frac{ib}{c} + p'.$$

Arrivals during this time are

$$\frac{1}{a}\left\{\frac{ib}{c} + p'\right\} = N_1' \text{ (say) or } N_1', \text{ the largest integer.}$$

The service time for these ships which will first wait would be given by

$$\frac{N_1' b}{c} + p_1'$$

and again during this time, the arrivals are

$$\frac{1}{a}\left\{\frac{N_1' b}{c} + p_1'\right\}.$$

The length of the line of ships will become zero after a lapse of time in which every ship that has waited has been served. Let A new arrivals have to wait in line, then certainly $(A + 1)$th arrival occurs after the service time of $i + A$ customers. i.e.

$$(i + A)\frac{b}{c} + p_2' \leqslant a(A + 1) \text{ giving } A = \frac{ib + c_2 p'_2 - ac}{ac - b}$$

Thus, the length of time required to serve all ships that have waited is approximately

$$b\left\{\frac{iac + cp'_2 - b}{ac - b}\right\} \qquad \ldots(20)$$

including the initial ship.

If the condition $a > \frac{b}{c} + k_m'$ is satisfied, then at least $a - \left(\frac{b}{c} + k_m'\right)$ time is saved for the service of every ship. Thus, the time saved up to the turn of the nth ship is at least

$$(n - 1)\left\{a - \left(\frac{b}{c} + k_m'\right)\right\}$$

before the nth ship is called on the service point.

The time required to serve i ships in the queue is $\frac{ib}{c} + p'$. If $(n_i' - 1)$ ships provide sufficient slack time, so that i ships may be served in it. Then

$$(n_i' - 1)\left\{a\left(\frac{b}{c} + k_m'\right)\right\} = i\,\frac{b}{c} + p'$$

giving

$$n_i' = \frac{c(a - k_m' + p') + b(i - 1)}{ac - b - ck_m'}.$$

Thus, after the end of time

$$T = b\,\frac{n_i'}{c} = \frac{b}{c}\left\{\frac{c(a - k_m' + p') + b(i - 1)}{ac - b - ck_m'}\right\}$$

the service facility will be free.

At time t, the number of ships that have entered the system is t/a, bringing the total to $\frac{t}{a} + i - 1$. During this time, $\frac{t}{\frac{b}{c} + k_m}$ ships would have been served. The number of ships in the system at time t would be $\frac{t}{a} + (i - 1) - \frac{tc}{b + ck_m}$.

If $W(t)$ is the waiting time for a ship in the system arriving after time t, then

$$W(t) = \begin{cases} 0 & \text{if } T - a \leqslant t. \\ t\left[\frac{\{b + ck_m - ac\}}{a(b + k_m)} + i\right]b - a & \text{if } 0 < t \leqslant T - a \\ (k - 1)\left(\frac{b}{c} + k_m\right) & \text{if } t = 0 \end{cases}$$

The total time spent in the system is given by (10)

REFERENCE

Saaty, T. L. (1961). Elements of Queueing Theory with Applications. McGraw-Hill Book Company, Inc., New York.