A NOTE ON CLT NUMBERS*

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The main purpose of this note is to answer a query by showing the existence of a natural number n such that every group of order n is supersoluble but not all groups of order n satisfy the converse of Lagrange's theorem.

1. INTRODUCTION

A natural number n is said to be a CLT number, if every group of order n is CLT, viz., if it has subgroups of all possible orders. It is said to be a supersoluble number if every group of order n is supersoluble. Since a supersoluble group is CLT [*see* e.g. McCarthy 1971, p. 589, (4.1)] it follows that a supersoluble number is a CLT number. The purpose of this note is to answer in the negative the query raised by McCarthy as to whether the converse is true.

The following notations will be adopted throughout :

Z_m = the cyclic group of order m

$|G|$ = the order of the group G

$[G:H]$ = the index of the subgroup H of G in G

$N_G(H)$ = the normalizer of the subgroup H of G in G

$C_G(H)$ = the centralizer of the subgroup H of G in G

$Z(G)$ = the centre of the group G

Aut G = the group of automorphisms of G

$A \times B$ = the direct product of the two groups A and B

ST = the product of the two subsets S and T of a group G

G/N = the factor group of the group G by the normal subgroup N of G

$\cong$ = isomorphic to

$\{1\}$ = identity subgroup

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2. The Main Theorem

The fact that a CLT number is not necessarily a supersoluble number is included in the following:

Theorem 1 — If p, q are primes (necessarily odd) such that $p \not\equiv 1 \pmod{q}$, $q^3 \equiv 1 \pmod{p}$, but $q, q^2 \not\equiv 1 \pmod{p}$, then pq^5 is a CLT number but not a supersoluble number.

Proof : Let, if possible, G be a non-CLT group of order pq^5. Since $p \not\equiv 1 \pmod{q}$, G has a normal Sylow q-subgroup Q. Let P be a Sylow p-subgroup of G. Then P is not normal in G, as otherwise $G = P \times Q$ is CLT, a contradiction. Since $q^r \equiv 1 \pmod{p} \Leftrightarrow r \equiv 0 \pmod{3}$, it follows that P has q^3 distinct conjugates in G so that $[G{:}N_G(P)] = q^3$. This means $|\,N_G(P)\,| = pq^2$. Let $N = Q \cap N_G(P)$. Then N is normal in $N_G(P)$ and $N_G(P) = P \times N$ is abelian. Hence, $C_G(N) \supseteq N_G(P)$. If R is a subgroup of G containing $N_G(P)$, then $N_G(P) = N_R(P)$, so that $[R{:}N_G(P)] = [R{:}N_R(P)]$, being equal to the number of distinct conjugates of P in R, cannot be q or q^2 as $q, q^2 \not\equiv 1 \pmod{p}$. It follows that $N_G(P)$ is a maximal subgroup of G. But $N_G(N) \supseteq C_G(N) \supseteq N_G(P)$ and $N_G(N) \supsetneq N_G(P)$, as N is not self-normalizing in Q. Hence, $N_G(N) = G$. But then $G/C_G(N)$ is isomorphic to a subgroup of Aut N whose order is not divisible by q^3 and so $C_G(N) = G$. Thus, $N \subseteq Z(G)$. Next, let, if possible, G have a subgroup T of order q^3 normalized by P. Then $K = PT$ is a subgroup of order pq^3 and P is not normal in K, since $|\,N_G(P)\,| = pq^2$. As $q, q^2 \not\equiv 1 \pmod{p}$, $[K{:}N_K(P)]$ cannot be q or q^2 and so $N_K(P) = P$. But N normalizes P. Hence, $N \cap K = N \cap P = \{1\}$. Consequently, $NK = G$ also and, in fact, $G = N \times K$, since $N \subseteq Z(G)$. But then G is CLT, as is easily verified, a contradiction. Thus, G has no subgroup of order q^3 normalized by P. This also means that Q has no characteristic subgroup of order q^3. Suppose Q is abelian. Noting that the Frattini subgroup of Q is characteristic in Q and that the set of all elements of order q together with identity form a characteristic subgroup of Q, it follows that either $Q \cong Z_{q^2} \times Z_q \times Z_q \times Z_q$ or Q must be elementary abelian. But then the number of elementary abelian subgroups of order q^3 contained in Q, being equal to $(q^4 - 1) \div (q - 1)$ in the former case and $(q^5 - 1)(q^4 - 1)(q^3 - 1) \div (q^3 - 1)(q^2 - 1)(q - 1)$ in the latter case (*see* e.g. Carmichael 1937, p. 118, Exercise 16), is not divisible by p, whereas p is the number of distinct conjugates (in G) of every elementary abelian subgroup of order q^3, an impossibility. Thus, Q is not abelian. $Q/Z(Q)$ is then non-cyclic and so $|\,Z(Q)\,| \neq q^4$. Since $N \subseteq Z(Q)$ and Q has no characteristic subgroup of order q^3, it follows that $N = Z(Q)$. Let S be a subgroup of order q contained in N. Then S is normal in G, since $N \subseteq Z(G)$. Suppose, if possible, Q/S is non-abelian. Then the possible orders for the centre and the commutator subgroup of Q/S are q, q^2. But at least one of them should be of order different from q (*see* e.g. Solution [1973, 83] of Problem 5815 [1971, 912] of the American Mathematical Monthly). Thus,

Q/S has a characteristic subgroup D/S of order q^2. D/S is then normal in G/S, so that D is normal in G and $|D| = q^3$, a contradiction. Thus, Q/S is abelian. As S is of prime order, it follows that the commutator subgroup of Q is cyclic and is contained in $N = Z(Q)$. But then $Q/Z(Q) \cong H \times H$ for some group H (*see* e.g. the theorem proved in the solution by C.R.B. Wright [1970, 1016] of Problem 5689 [1969, 947] of the American Mathematical Monthly), an impossibility, since $|Q/Z(Q)| = |Q/N| = q^3 =$ an odd power of q. This establishes the non-existence of G. A non-supersoluble group of order pq^5 is given by the direct product of $Z_q \times Z_q$ and the relative holomorph of the elementary abelian group of order q^3 with Z_p. Or, alternatively, the existence of a non-supersoluble group of order pq^5 follows from Theorem 1.5 of Pazderski (1959).

Corollary 1 — If p, q are primes such that $p \not\equiv 1 \pmod q$, $q^3 \equiv 1 \pmod p$, but $q, q^2 \not\equiv 1 \pmod p$, then every non-nilpotent group of order pq^5 is the direct product of a group of order q^2 and a non-CLT group of order pq^3 whose Sylow q-subgroup is elementary abelian.

PROOF : Let G be a non-nilpotent group of order pq^5, where p, q satisfy the conditions stated above. Then by the above theorem, G has a subgroup K of order pq^3. Let P be a Sylow p subgroup of K, which is also a Sylow p-subgroup of G and let Q be the Sylow q-subgroup of G. Then the proof of the above theorem shows that $G = N \times K$, where $N = Q \cap N_G(P)$. K has no subgroup of order pq, because, otherwise, K would have a subgroup of order q normalizing P, whereas P is self-normalizing in K by virtue of P being non-normal in K and $q, q^2 \not\equiv 1 \pmod p$. The Sylow q-subgroup $Q \cap K$ of K is elementary abelian, because it has no characteristic subgroup of order q, as K has no subgroup of order q normalized by P.

Corollary 2 — If p, q are primes such that $p \not\equiv 1 \pmod q$, $q^3 \equiv 1 \pmod p$, but $q, q^2 \not\equiv 1 \pmod p$, then the only groups of order q^5 having an automorphism of order p are $Z_{q^2} \times Z_q \times Z_q \times Z_q$ and the elementary abelian group.

The proof follows from Corollary 1 once we note that a group of order q^5 can be the Sylow q-subgroup of a non-nilpotent group of order pq^5 if and only if it has an automorphism of order p.

After Theorem 1, it is meaningful to characterize all CLT numbers, since such a characterization is not immediate from the characterization of supersoluble numbers given by Pazderski (1959). As a first step, it is natural to enquire whether there are subclasses of natural numbers other than the one specified in Theorem 1 consisting only of CLT numbers which are not supersoluble numbers. In the first instance, one may be tempted to conjecture that pq^{2r-1}, where p, q are primes, such that $p \not\equiv 1 \pmod q$ and $\mathrm{Ord}_p(q) = r > 2$ [here and elsewhere $\mathrm{Ord}_p(q) = r$ means $q^r \equiv 1 \pmod p$, but $q^s \not\equiv 1 \pmod p$ for any s with $1 \leqslant s < r$] is a CLT number. The conjecture is generally false, as shown below.

Theorem 2 — For any two odd primes p, q, such that $\mathrm{Ord}_p(q)$ is an even positive integer r, there exists a group of order pq^{2r-1} not having a subgroup of order pq^r.

PROOF : Let $r = 2n$ and Q_1 be the extra-special q-group of order q^{r+1} and exponent q obtained by taking the central product of n copies of the non-abelian group of order q^3 and exponent q. Then Q_1 has an automorphism of order $q^n + 1$ [*see* e. g. Gorenstein 1968, p. 215, Chapter 5, Exercise 18]. Since p divides $q^r - 1 = q^{2n} - 1$ and p does not divide $q^n - 1$, it follows that p divides $q^n + 1$ and so Q_1 has an automorphism of order p. Thus, there exists a proper semi-direct product A of Q_1 with a cyclic group P of order p. Since $[A:N_A(P)] \equiv 1 \pmod p$ and $\mathrm{Ord}_p(q)$ equals r, it follows that $N_A(P)$ is a subgroup of order pq. On the other hand, $Z(Q_1)$ is a subgroup of order q, which is characteristic in Q_1 and hence normal in A, so that $PZ(Q_1)$ is a subgroup of order pq in which P is also normal, since $q \not\equiv 1 \pmod p$. Hence, $N_A(P) = PZ(Q_1) = P \times Z(Q_1)$ and $N_A(P)$ is a maximal subgroup of A, since for any subgroup R of A containing $N_A(P)$, we have $[R:N_A(P)] = [R:N_R(P)] \equiv 1 \pmod p$, whereas $\mathrm{Ord}_p(q) = r$. Now $Z(Q_1)$ centralizes both P and Q_1 and hence their product A. In fact, $Z(A) = Z(Q_1)$, since $| Z(A) |$ cannot be divisible by p, as P does not centralize A. Thus, $| Z(A) | = q$, so that there exists an isomorphism of $Z(A)$ into the centre of any group N of order q^{r-1}. Hence, there exists a group G, the central product of A and N, such that $G = AN$ with $Z(A) = A \cap N \subseteq Z(G)$ and A centralizes N [*see* e.g. Gorenstein 1968, p. 29, Chapter 2, Theorem 5.3]. Clearly, $| G | = | A | \; | N | \div | A \cap N | = pq^{r+1}q^{r-1} \div q = pq^{2r-1}$. Let, if possible, G have a subgroup of order pq^r. Then it is easily seen that G has a subgroup T of order q^r normalized by P. Since Q_1 is also normalized by P, it follows that $S = T \cap Q_1$ is a subgroup of order $| T | \; | Q_1 | \div | TQ_1 | \geqslant q^r q^{r+1} \div q^{2r-1} = q^2$ normalized by P, noting that TQ_1 is a subset of the Sylow q-subgroup of G, so that $| TQ_1 | \leqslant q^{2r-1}$. Then PS is a subgroup of order pq^s with $2 \leqslant s \leqslant r$. If $s < r$, then P is normal in PS and consequently normal in A (since $\mathrm{Ord}_p(q) = r > s \geqslant 2$), a contradiction. So let $s = r$. Then $| S | = p^r = | T |$, so that $S = T$, as $S \subseteq T$. Hence, $T \subseteq Q_1$. Further, T is maximal and hence normal in Q_1, so that $T \cap Z(Q_1) \neq \{1\}$. As $| Z(Q_1) | = q$, it follows that $T \supseteq Z(Q_1)$. Hence, $A = PQ_1$ contains a subgroup PT of order pq^r containing $PZ(Q_1) = N_A(P)$, a contradiction to the already observed fact that $N_A(P)$ is a maximal subgroup of A.

3. CONCLUDING REMARKS

Assuming that the primes p, q are odd, it is, of course, possible to delete (at the cost of a longer proof) the hypothesis $p \not\equiv 1 \pmod q$ in Theorem 1, but this is not needed for the main purpose of this note. Again, in unpublished results communicated to the author by Prof. Ruth Rebekka Struik, are found the following, relating to Theorem 2 : (i) If p, q are odd primes such that $\mathrm{Ord}_p(q)$ is an odd positive, integer r, then pq^{2r-1} is a CLT number. (ii) There is no non-CLT group of order pq^{2r-1} with abelian normal Sylow q-subgroup, if p, q are primes such that $\mathrm{Ord}_p(q) = r$.

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References

Carmichael, R. D. (1937). Introduction to the Theory of Groups of Finite Order. Dover Publs., New York.

Gorenstein, D. (1968). Finite Groups. Harper & Row.

McCarthy, D. J. (1971). A survey of partial converses to Lagrange's Theorem on finite groups. *Trans. N. Y. Acad. Sci.*, **33**, 586-94.

Pazderski, G. (1959). Die Ordnung zu denen nur Gruppen mit gegebener Eigenschaft gehören. *Arch. Math.*, **10**, 331-43.