SOME APPLICATIONS OF TRIPLE INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS

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Some three-part mixed boundary value problems of the mathematical theory of elasticity are solved by reducing them to triple integral equations involving inverse Mellin transforms. These equations are solved by the method of Tweed, which involves reduction of these equations to a Fredholm integral equation of the second kind. The Fredholm equation is then solved iteratively and finally the quantities of physical interest are found.

1. INTRODUCTION

Recently, Tweed (1973) reported an elegant method of solving triple integral equations involving inverse Mellin transforms. He also illustrated the use of these equations by considering their application to a few problems of the mathematical theory of elasticity. Here, we shall further illustrate the use of these equations by considering their application to a few more complicated problems of the mathematical theory of elasticity. The following two problems are considered.

Problem 1 : *Stress distribution in a thin circular disc containing two Griffith cracks in the region* $a < r < 1, \theta = 0, \pi$

The material of the disc is supposed to be homogeneous and isotropic with modulus of rigidity μ and Poisson's ratio η The cracks are opened by applying normal pressure $f(r)$ to its faces. It may be observed that four quadrants are symmetrical. For a symmetrical deformation, the displacement vector U will have components u_r, u_θ and 0 and the stress tensor will have the components $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and σ_{rr}. We shall also assume that the circular boundary of the disc is stress free.

Problem 2 : *External cracks*

We shall determine the stress distribution in a two-dimensional medium containing a circular hole and two symmetrically situated Griffith cracks. The two cracks occupy the region $a < r < 1$, $\theta = 0$, π and the circular hole the region $0 \leqslant r \leqslant \rho$ $(\rho < a)$, $0 \leqslant \theta \leqslant 2\pi$. The medium is homogeneous and isotropic with modulus of rigidity μ and Poisson's ratio η. The cracks are opened by the internal normal pressure $f(r)$.

It will be observed that in these problems, there is symmetry in the four quadrants. We shall also assume that the circular hole is stress-free.

2. Components of Stresses and Displacements

The components of stresses and displacements for an infinite elastic medium, as given by Tweed (1973), are

$$\sigma'_{\theta\theta}(r, \theta) = \frac{1}{r^2} M^{-1}[s(s+1), \bar{\psi}(s, \theta); r] \qquad ...(2.1)$$

$$\sigma'_{r\theta}(r, \theta) = \frac{1}{r^2} M^{-1}\left[s(s+1)\frac{d\bar{\psi}(s, \theta)}{d\theta}; r\right] \qquad ...(2.2)$$

$$\sigma'_{rr}(r, \theta) = \frac{1}{r^2} M^{-1}\left[\frac{d^2\bar{\psi}(s, \theta)}{d\theta^2} - s\bar{\psi}(s, \theta); r\right] \qquad ...(2.3)$$

$$2\mu u'_{\theta}(r, \theta) = \frac{1}{r} M^{-1}\Bigg[\frac{-1}{(s+1)(s+2)}\Big\{(1-\eta)\frac{d^3\bar{\psi}(s, \theta)}{d\theta^3} + [(1-\eta)s^2 + (s+1)(s+2)]\frac{d\bar{\psi}(s, \theta)}{d\theta}\Big\}; r\Bigg] \qquad ...(2.4)$$

$$2\mu u'_{r}(r, \theta) = \frac{1}{r} M^{-1}\Bigg[\frac{1}{(s+1)}\Big\{(1-\eta)\frac{d^2\bar{\psi}(s, \theta)}{d\theta^2} - s(1+s\eta)\bar{\psi}(s, \theta)\Big\}; r\Bigg] \qquad ...(2.5)$$

where M^{-1} is the inverse Mellin transform and

$$\bar{\psi}(s, \theta) = \int_0^\infty \psi(r, \theta)\, r^{s-1}\, dr.$$

For the two problems, the appropriate expressions for the stress and displacement components are obtained by superimposing the solutions for the circular plate and the circular hole respectively to the above solutions. Hence, the displacement and stress components for the two problems are :

Problem 1

$$\sigma_{\theta\theta}(r, \theta) = \sigma'_{\theta\theta}(r, \theta) + \sum_{n=0}^{\infty} [n(n-1)\, a_n\, r^{n-2} + (n+1)(n+2)\, b_n \,.\, r^n] \cos n\alpha \qquad ...(2.6)$$

$$\sigma_{r\theta}(r, \theta) = \sigma'_{r\theta}(r, \theta) + \sum_{n=0}^{\infty} [n(n-1)\, a_n \,.\, r^{n-2} + n(n+1)\, b_n r^n] \sin n\alpha \qquad ...(2.7)$$

$$\sigma_{rr}(r, \theta) = \sigma'_{rr}(r, \theta) - \sum_{n=0}^{\infty} [n(n-1) a_n \cdot r^{n-2}$$

$$+ (n+1)(n-2) b_n r^n] \cos n\alpha \quad \text{...(2.8)}$$

$$2\mu u_\theta(r, \theta) = 2\mu u'_\theta(r, \theta) + \sum_{n=0}^{\infty} [n\, a_n r^{n-1}$$

$$+ (n + 4 - 4\eta) b_n r^{n+1}] \sin n\alpha \quad \text{...(2.9)}$$

$$2\mu u_r(r, \theta) = 2\mu u'_r(r, \theta) - \sum_{n=0}^{\infty} [n\, a_n r^{n-1}$$

$$+ (n - 2 + 4\eta) b_n r^{n+1}] \cos n\alpha. \quad \text{...(2.10)}$$

Problem 2

$$\sigma_{\theta\theta}(r, \theta) = \sigma'_{\theta\theta}(r, \theta) - \sum_{n=0}^{\infty} [\{(n+2) a_n + b_n\} \cos n\alpha\, r^{-n-2}$$

$$- n a_n r^{-n-2} \cos (n+2)\alpha] \quad \text{...(2.11)}$$

$$\sigma_{r\theta}(r, \theta) = \sigma'_{r\theta}(r, \theta) + \sum_{n=0}^{\infty} [\{(n+2) a_n + b_n\} r^{-n-2}$$

$$\times \sin n\alpha - (n+2) a_n r^{-n-2} \sin (n+2)\alpha] \quad \text{...(2.12)}$$

$$\sigma_{rr}(r, \theta) = \sigma'_{rr}(r, \theta) + \sum_{n=0}^{\infty} [\{(n+2) a_n + b_n\} r^{-n-2}$$

$$\times \cos n\alpha - (n+4) a_n r^{-n-2} \cos (n+2)\alpha] \quad \text{...(2.13)}$$

$$2\mu u_\theta(r, \theta) = 2\mu u'_\theta(r, \theta) - \sum_{n=0}^{\infty} \left[\left\{\frac{(n+2) a_n + b_n}{n+1}\right\} r^{-n-1}\right.$$

$$\left. \times \sin n\alpha - \frac{n + 4\eta - 2}{n+1} a_n r^{-n-1} \sin (n+2)\alpha \right] \quad \text{...(2.14)}$$

$$2\mu u_r(r, \theta) = 2\mu u'_r(r, \theta) - \sum_{n=0}^{\infty} \left[\left\{\frac{(n+2) a_n + b_n}{n+1}\right\} r^{-n-1}\right.$$

$$\left. \times \cos n\alpha - \frac{n + 4 - 4\eta}{n+1} a_n r^{-n-1} \cos (n+2)\alpha \right] \quad \text{...(2.15)}$$

where $\alpha = \theta - \frac{\pi}{2}$.

3. Boundary Conditions and Derivation of Triple Integral Equations

Problem 1

The boundary conditions to be satisfied on the line $\theta = 0$ are

$$\sigma_{\theta\theta}(r, 0) = \frac{f(r)}{\pi}, \quad a < r < 1 \qquad \ldots(3.1)$$

$$\sigma_{r\theta}(r, 0) = 0, \quad 0 \leqslant r \leqslant \rho \qquad \ldots(3.2)$$

$$u_\theta(r, 0) = 0, \quad (0 \leqslant r < a \text{ and } 1 < r \leqslant \rho). \qquad \ldots(3.3)$$

Since the circular boundary of the disc is assumed to be stress-free, we have, in addition to the above

$$\sigma_{r\theta}(\rho, \theta) = \sigma_{rr}(\rho, \theta) = 0, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}. \qquad \ldots(3.4)$$

The appropriate expression for the Airy's stress function $\bar{\psi}(s, \theta)$ for the two problems is

$$\bar{\psi}(s, \theta) = A(s) \left[\frac{(s+2)\cos s\alpha \sin (s+2)\frac{\pi}{2} - s\cos(s+2)\alpha \sin s\frac{\pi}{2}}{4s(s+1)\sin s\frac{\pi}{2}\sin(s+2)\frac{\pi}{2}} \right] \qquad \ldots(3.5)$$

where $A(s)$ is an unknown function to be determined later.

Substituting the value of $\bar{\psi}(s, \theta)$ in (2.7), we find that eqn. (3.2) is automatically satisfied provided the odd coefficients a_{2n+1} and b_{2n+1} are zero. The boundary conditions (3.1) and (3.3) are satisfied provided the unknown function $A(s)$ satisfies the triple integral equations

$$M^{-1}\left[\cot s\frac{\pi}{2} A(s); r\right] = F(r), \quad a < r < 1 \qquad \ldots(3.6)$$

$$M^{-1}\left[(1+s)^{-1} A(s); r\right] = 0, \quad (0 \leqslant r < a \text{ and } 1 < r \leqslant \rho) \qquad \ldots(3.7)$$

where

$$F(r) = \left(\frac{2r^2}{\pi}\right) f(r) - 2\sum_{n=0}^{\infty} (-1)^n (b_{2n} - a_{2n+2})\, r^{2n+2}. \qquad \ldots(3.8)$$

The solution of these equations, as given by Tweed (1973), is

$$A(s) = \int_a^1 p(t^2)\, t^{s+1}\, dt \qquad \ldots(3.9)$$

where $p(t^2)$ is an unknown function satisfying the condition

$$\int_a^1 p(t^2)\ dt = 0 \qquad \text{...(3.10)}$$

and is determined from the equation

$$p(t^2) = -\frac{2}{\pi} H[F(r)] + \frac{C}{T(t^2)} \qquad \text{...(3.11)}$$

where

$$H[F(r)] = \left(\frac{t^2 - a^2}{1 - t^2}\right)^{1/2} \int_a^1 \left(\frac{1 - r^2}{r^2 - a^2}\right)^{1/2} \frac{F(r)\, dr}{(r^2 - t^2)}$$

and $T(t^2) = \sqrt{\{(t^2 - a^2)(1 - t^2)\}}$

Problem 2

The boundary conditions to be satisfied on the line $\theta = 0$ are :

$$\sigma_{\theta\theta}(r, 0) = \frac{f(r)}{\pi},\ a < r < 1 \qquad \text{...(3.12)}$$

$$\sigma_{r\theta}(r, 0) = 0,\ \rho \leqslant r < \infty \qquad \text{...(3.13)}$$

$$u_\theta(r, 0) = 0,\ (\rho \leqslant r < a \text{ and } r > 1). \qquad \text{...(3.14)}$$

Since the circular hole is stress-free, we have

$$\sigma_{r\theta}(\rho, \theta) = \sigma_{rr}(\rho, \theta) = 0,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}. \qquad \text{...(3.15)}$$

For this problem also, the Airy's stress function is given by (3.5). The boundary condition (3.13) is satisfied provided the odd coefficients a_{2n+1} and b_{2n+1} are all zero. The boundary conditions (3.12) and (3.14) lead to the triple integral equations for the determination of the unknown function $A(s)$

$$M^{-1}\left[\cot\frac{s\pi}{2} A(s); r\right] = G(r),\ a < r < 1 \qquad \text{...(3.16)}$$

$$M^{-1}[(1 + s)^{-1} A(s); r] = 0,\ (\rho \leqslant r < a \text{ and } r > 1) \qquad \text{...(3.17)}$$

where

$$G(r) = \left(\frac{2r^2}{\pi}\right) f(r) + 2\sum_{n=0}^{\infty} (-1)^n [2(2n + 1)\, a_{2n} + b_{2n}]\, r^{-2n}. \qquad \text{...(3.18)}$$

The solution of this set of equations is

$$A(s) = \int_a^1 p(t^2)\, t^{s+1}\, dt \qquad \text{...(3.19)}$$

where $p(t^2)$ is determined from the equation

$$p(t^2) = -\frac{2}{\pi} H[G(r)] + \frac{C}{T(t^2)} \qquad \text{...(3.20)}$$

and the condition (3.10).

4. Conditions on the Free Boundary

Problem 1

We shall now determine the unknown coefficients a_n and b_n by applying the conditions (3.4). Substituting the value of $\bar{\psi}(s, \theta)$ from (3.5) in (2.7) and (2.8) and using the results of the appendix, we get

$$\int_a^1 p(t^2)\, t \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi} \left\{n - (n+1)\left(\frac{t}{\rho}\right)^2\right\}\left(\frac{t}{\rho}\right)^{2n-2} \sin 2n\alpha\right] dt$$

$$+ \sum_{n=0}^{\infty} [2n(2n-1)\, a_{2n}\, \rho^{2n} + 2n(2n+1)\, b_{2n}\, \rho^{2n+2}] \sin 2n\alpha = 0 \qquad \text{...(4.1)}$$

$$\int_a^1 p(t^2)\, t \left[\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi} (n+1) \left\{1 - \left(\frac{t}{\rho}\right)^2\right\}\left(\frac{t}{\rho}\right)^{2n-2} \cos 2n\alpha\right] dt$$

$$+ \sum_{n=0}^{\infty} [2n(2n-1)\, a_{2n} \,.\, \rho^{2n} + (2n+1)(2n-2)\, b_{2n}\, \rho^{2n+2}] \cos 2n\alpha = 0. \qquad \text{...(4.2)}$$

From these, we derive

$$2b_0\rho^2 = \frac{1}{\pi} \int_a^1 tp(t^2)\, dt \qquad \text{...(4.3)}$$

and for $n \geqslant 1$

$$2n(2n-1)\, a_{2n}\, \rho^{2n} = \frac{(-1)^n}{\pi} \int_a^1 tp(t^2).$$

$$\times \left[2n^2 \left(\frac{t}{\rho}\right)^{2n-2} - (2n-1)(n+1)\left(\frac{t}{\rho}\right)^{2n}\right] dt \qquad \text{...(4.4)}$$

$$2.(2n+1)\,b_{2n}\,\rho^{2n+2} = \frac{(-1)^n}{\pi}\int_a^1 tp(t^2)\left[(2n+2)\left(\frac{t}{\rho}\right)^{2n} - (2n+1)\left(\frac{t}{\rho}\right)^{2n-2}\right]dt. \quad \text{...(4.5)}$$

Substituting these coefficients in (3.8), we obtain

$$F(r) = \frac{2r^2}{\pi} f(r) - \frac{2}{\pi}\int_a^1 tp(t^2)\,K(t^2, r^2)\,dt \quad \text{...(4.6)}$$

where

$$K(t^2, r^2) = \frac{3r^2}{\rho^2} - \frac{2t^2r^2}{\rho^4} - \sum_{n=1}^{\infty}\left(\frac{r}{t}\right)^2\left(\frac{tr}{\rho^2}\right)^{2n}$$
$$\left[(n+1)(2n+1) - 4(n+1)^2\left(\frac{t}{\rho}\right)^2 + (n+1)(n+2)\left(\frac{t}{\rho}\right)^4\right].$$

Now integrating eqn. (3.11) with respect to t from a to 1 and utilizing condition (3.10), we obtain

$$C = \frac{2}{\pi F}\int_a^1 H[F(r)]\,dt$$

where

$$F = F\left(\frac{\pi}{2}, q\right); \quad q = \sqrt{(1-a^2)}$$

is the complete elliptic integral of the first kind, and

$$p(t^2) = -\frac{2}{\pi} H[F(r)] + \frac{2}{\pi F T(t^2)}\int_a^1 H[F(r)]\,dt. \quad \text{...(4.7)}$$

Substitution for $F(r)$ from (4.6) in (4.7) shows that $p(t^2)$ is the solution of the Fredholm integral equation of the second kind.

$$p(t^2) = \phi(t^2) + \int_a^1 xp(x^2)\,M(x^2, t^2)\,dx \quad \text{...(4.8)}$$

where

$$\phi(t^2) = \frac{4}{\pi^2} H[r^2 f(r)] + \frac{4}{\pi 2 F T(t^2)}\int_a^1 H[(r^2 f(r)]\,dt.$$

$$M(x^2, t^2) = \frac{4}{\pi^2} H[K(x^2, r^2)] - \frac{4}{\pi 2FT(t^2)} \int_a^1 H[K(x^2, r^2)]\, dt.$$

In the case when $\rho > 1$, the function $K(x^2, r^2)$ can be approximated to

$$K(x^2, r^2) = \frac{3r^2}{\rho^2} - \frac{2r^2}{\rho^4}(x^2 + 3r^2) + \frac{16}{\rho^6} x^2 r^4 + o(\rho^{-8}) \qquad \ldots(4.9)$$

so that

$$M(x^2, t^2) = \frac{2}{\pi T(t^2)} \Big[\Big(t^2 - \frac{E}{F}\Big)(3\rho^{-2} - 2x^2\rho^{-4}) - (3\rho^{-4} - 8x^2\rho^{-6}) \times \{2t^4 - (a^2 + 1)\, t^2 + \beta\} + o(\rho^{-8}) \Big] \qquad \ldots(4.10)$$

where

$$\beta = \tfrac{1}{3}\Big[2a^2 - (a^2 + 1)\frac{E}{F}\Big]$$

and $E = E\Big(\frac{\pi}{2}, q\Big)$, $F = F\Big(\frac{\pi}{2}, q\Big)$

are the complete elliptic integrals of the first and second kind.

It should be noticed that the kernel $M(x^2, t^2)$ has weak singularities, which can be removed by suitable change of variable and hence all the fundamental Fredholm theorems can be applied to eqn. (4.8). An iterative solution of (4.8) can be obtained by writing

$$p(t^2) = p_0(t^2) + \rho^{-1} p_1(t^2) + \rho^{-2} p_2(t^2) + \ldots \qquad \ldots(4.11)$$

so that

$$p_0(t^2) = \phi(t^2), \quad p_1(t^2) = p_3(t^2) \ldots = 0$$

$$p_2(t^2) = \frac{6\Big(t^2 - \frac{E}{F}\Big)}{\pi T(t^2)} \int_a^1 x p_0(x^2)\, dx$$

$$p_4(t^2) = \frac{-2}{\pi T(t^2)} \Big[\Big(t^2 - \frac{E}{F}\Big) \int_a^1 \{2x^2 p_0(x^2) - 3p_2(x^2)\}\, x\, dx + 3\{2t^4 - (a^2 + 1)\, t^2 + \beta\} \int_a^1 x p_0(x^2)\, dx \Big]$$

$$p_6(t^2) = \frac{2}{\pi T(t^2)}\left[\left(t^2 - \frac{E}{F}\right)\int_a^1 \{3p_4(x^2) - 2x^2 p_2(x^2)\}\, x\, dx\right.$$

$$\left. + \{2t^4 - (a^2+1)\, t^2 + \beta\}\int_a^1 \{8x^2 p_0(x^2) - 3p_2(x^2)\}\, x\, dx.\right]$$

The case of most direct physical importance is that when the cracks are opened by a constant normal pressure p_0. In this case, we find

$$p(t^2) = \frac{2p_0}{\pi T(t^2)}\left[\left(t^2 - \frac{E}{F}\right)(1 + c_0\rho^{-2} + c_1\rho^{-4} + c_2\rho^{-6})\right.$$

$$\left. + \{2t^4 - (a^2+1)\, t^2 + \beta\}(c_2\rho^{-4} + c_4\rho^{-6}) + o(\rho^{-8})\right] \quad \ldots(4.12)$$

where

$$c_0 = \tfrac{3}{2}\left(a^2 + 1 - \frac{2E}{F}\right),$$

$$c_1 = c_0^2 - \left\{\frac{c_0(a^2+1)}{3} - \left(\frac{a^2-1}{2}\right)^2\right\}$$

$$c_2 = -\, c_0$$

$$c_3 = c_0\{c_1 - 2(c_1 - c_0)^2\}$$

$$c_4 = 2c_0c_2 - 8(c_1 - c_0^2).$$

Problem 2

Proceeding in the same manner as in the case of problem 1, we find that $p(t^2)$ is given by

$$p(t^2) = \frac{p_0}{2\pi T(t^2)}\left[\left(t^2 - \frac{E}{F}\right) + (c_0\rho^2 + c_1\rho^4 + c_3\rho^6)\right.$$

$$\left. \times\left(t^{-2} - \frac{E}{a^2F}\right) + c_2\rho^6(\beta + 2t^{-2} - 2t^{-4}) + o(\rho^8)\right] \quad \ldots(4.15)$$

where

$$c_0 = \left(a - \frac{E}{F}\right),\ c_1 = \frac{c_0}{2a^2}\left(1 + a^2 - \frac{E}{F}\right)$$

$$c_2 = \frac{E(1 + a^{-2})}{F} - 2,\ c_3 = \frac{c_1^2}{c_0} + (a^{-2} - 1)\, c_2.$$

5. Stress Intensity Factors and Shape of Deformed Cracks

Problem 1

The stress intensity factors at two ends of the crack are given by

$$N_a = \mathop{\mathrm{Lt.}}_{r \to a^-} (a - r)^{1/2} [\sigma_{\theta\theta}(r, 0)], \quad 0 < r < a \qquad \ldots(5.1)$$

$$N_1 = \mathop{\mathrm{Lt.}}_{r \to 1^+} (r - 1)^{1/2} [\sigma_{\theta\theta}(r, 0)], \quad r > 1 \qquad \ldots(5.2)$$

From (2.6), we observe that

$$\sigma_{\theta\theta}(r, 0) = \frac{1}{2r} M^{-1}\left[\cot \frac{s\pi}{2} A(s) : r\right] + \sum_{n=0}^{\infty} (-1)^n \times (2n + 1)(2n + 2)(b_{2n} - a_{2n+2}) r^{2n}.$$

Substituting for $A(s)$ from (3.9) and the values of a_{2n} and b_{2n} from (4.3) to (4.5), we get

$$\sigma_{\theta\theta}(r, 0) = \frac{1}{\pi} \int_a^1 \frac{tp(t^2)}{t^2 - r^2} dt + \frac{1}{\pi^2} \int_a^1 \frac{K(t^2, r^2)}{r^2} tp(t^2) dt \qquad \ldots(5.3)$$

where $K(t^2, r^2)$ is given by (4.9). Replacing $p(t^2)$ from (4.12), we observe that for $0 < r < a$

$$\frac{1}{\pi} \int_a^1 \frac{tp(t^2)}{t^2 - r^2} dt = \frac{p_0}{\pi} \Bigg[\left(1 + \frac{r^2 - (E/F)}{R}\right)(1 + c_0 \rho^{-2} + c_1 \rho^{-4} + c_3 \rho^{-6}) + \left\{2r^2 + \frac{2r^4 - (a^2 + 1) r^2 + \beta}{R}\right\} \times (c_2 \rho^{-4} + c_4 \rho^{-6}) + o(\rho^{-8})\Bigg]$$

where $R = \{(a^2 - r^2)(1 - r^2)\}^{1/2}$ and for $r > 1$, we have

$$\frac{1}{\pi} \int_a^1 \frac{tp(t^2)}{t^2 - r^2} dt = \frac{p_0}{\pi} \Bigg[\left(1 + \frac{(E/F) - r^2}{R_1}\right) \times (1 + c_0 \rho^{-2} + c_1 \rho^{-4} + c_3 \rho^{-6}) + \left\{2r^2 - \frac{2r^4 - (a^2 + 1) r^2 + \beta}{R_1}\right\} \times (c_2 \rho^{-4} + c_4 . \rho^{-6}) + o(\rho^{-8})\Bigg]$$

where $R_1 = \sqrt{\{(r^2 - a^2)(r^2 - 1)\}}$. It is easy to show that

$$\int_a^1 \frac{K(t^2, r^2)}{r^2} tp(t^2)\, dt = \frac{p_0}{\pi} [c_0 \rho^{-2} + \rho^{-4} \{c_0^2 + c_0(c_1 - c_0^2) - \tfrac{2}{3} c_0 r^2\}$$

$$- \rho^{-6} \{\tfrac{2}{3} c_0^2 + 8(c_1 - c_0^2)\} + o(\rho^{-8})].$$

Hence, the stress intensity factors are

$$N_a = \frac{p_0}{\pi\{2a(1 - a^2)\}^{1/2}} \left[\left(a^2 - \frac{E}{F}\right)(1 + c_0 \rho^{-2} + c_1 \rho^{-4} + c_3 \rho^{-6}) + \{a^2(a^2 - 1) + \beta\}(c_2 \rho^{-4} + c_4 \rho^{-6}) + o(\rho^{-8})\right] \quad \ldots(5.4)$$

$$N_1 = \frac{p_0}{\pi\{2(1 - a^2)\}^{1/2}} \left[\left(\frac{E}{F} - 1\right)(1 + c_0 \rho^{-2} + c_1 \rho^{-4} + c_3 \rho^{-6}) + (a^2 - 1 - \beta)(c_2 \rho^{-4} + c_4 \rho^{-6}) + o(\rho^{-8})\right]. \quad \ldots(5.5)$$

The normal displacement along the surface of the crack is given by

$$2\mu u_\theta(r, 0) = -\frac{(1 - \eta)}{r} M^{-1} [(1 + s)^{-1} A(s) : r]$$

$$= -(1 - \eta) \int_r^1 p(t^2)\, dt, \quad a < r < 1. \quad \ldots(5.6)$$

Hence

$$\frac{\pi\mu u_\theta(r, 0)}{p_0(\eta - 1)} = [1 + c_0 \rho^{-2} + \{c_1 + \tfrac{1}{3}(a^2 + 1) c_2\} \rho^{-4} + \{c_3 + \tfrac{1}{3}(a^2 + 1) c_4\} \rho^{-6}]\left[E(\phi, q)\frac{E}{F} - F(\phi, q)\right] + \frac{2r}{3}\{(r^2 - a^2)(1 - r^2)\}^{1/2}(c_2 \rho^{-4} + c_4 \rho^{-6}) + o(\rho^{-8})\Big] \quad \ldots(5.7)$$

where $\sin\phi = \left\{\frac{(1 - r^2)}{(1 - a^2)}\right\}^{1/2}$.

Numerical calculations were carried out for radius of the disc $\rho = 2$ units of lengths, and for $a = 0.5$, 0.259 and 0.0175. The curves for normal displacement of cracks are shown in Fig. 1. The continuous curves correspond to the problem under discussion and the dotted curves correspond to the problem considered by Tranter (1961). It is obvious that the opening is more compared to Tranter's problem. Also, the stress concentration at origin is more in our case.

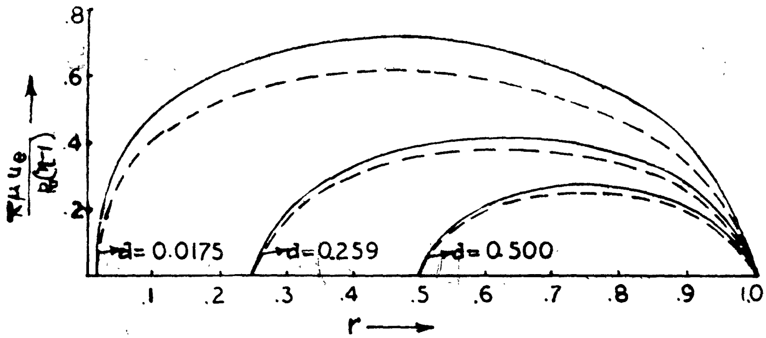

FIG. 1. Variation of normal displacement with r and a.

Problem 2

We shall now briefly write the expressions for stress intensity factors and normal displacement of crack faces in this case. The stress intensity factors in this case are

$$N_a = \frac{p_0}{\pi\{2a(1-a^2)\}^{1/2}}\left[\left(a^2 - \frac{E}{F}\right) + \left(1 - \frac{E}{F}\right)a^{-2}(c_0\rho^2 + c_1\rho^4 + c_3\rho^6) + c_2 a^{-4}\rho^6(a^4\beta + 2a^2 - 2) + o(\rho^8)\right] \quad ...(5.8)$$

$$N_1 = \frac{p_0}{\pi\{2(1-a^2)\}^{1/2}}\left[\left(\frac{E}{F} - 1\right) + \left(\frac{E}{a^2F} - 1\right)(c_0\rho^2 + c_1\rho^4 + c_3\rho^6) - 2c_2\beta\rho^6 + o(\rho^8)\right] \quad ...(5.9)$$

The normal component of displacement is given by

$$\frac{4\pi\mu u_\theta(r,0)}{p_0(\eta - 1)} = \left[E(\phi, q) - \frac{E}{F}F(\phi, q)\right] \times [1 + a^{-2}(c_0\rho^2 + c_1\rho^4 + c_3\rho^6)] + a^{-2}c_2\rho^6\,[\tfrac{2}{3}(1 - 2a^{-2})\,E(\phi, q) + (\tfrac{2}{3} + a^2\beta)\,F(\phi, q) + \tfrac{1}{3}r^{-3}\{(r^2 - a^2)\,.\,(1 - r^2)\}^{1/2}] + o(\rho^8). \quad ...(5.10)$$

The displacement is numerically calculated taking the radius of the circular hole, $\rho = 0.20$ unit of length, and $a = 0.5$ and 0.259. The opening of the cracks is shown in Fig. 2. The continuous curves correspond to this problem, while the dotted curves correspond to Tranter's problem (1961).

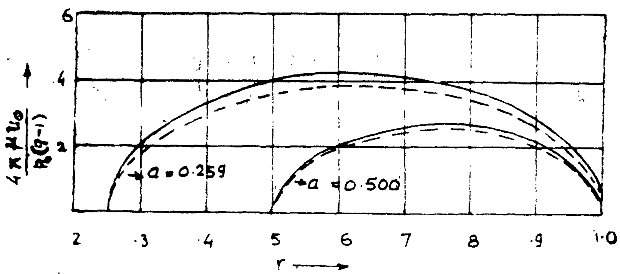

FIG. 2. Variation of normal displacement with r and a.

REFERENCES

Tranter, C. J. (1961). The opening of a pair of coplanar Griffith cracks under internal pressure. *Q. Jl Mech. appl. Math.*, **14**, No. 3.

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APPENDIX

Here, we shall evaluate some integrals which are used in Section 4. These integrals are evaluated making use of the theory of calculus of residues and the contour integration. We have for $t < \rho$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+2)\frac{\sin(s+2)\alpha}{\sin(s+2)\frac{\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds=\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi}4n\left(\frac{t}{\rho}\right)^{2n-2}\times\sin 2n\alpha \qquad \ldots(1)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+2)\frac{\sin s\alpha}{\sin\frac{s\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds=\sum_{n=0}^{\infty}\frac{2(-1)^n}{\pi}(2n+2)\left(\frac{t}{\rho}\right)^{2n}\times\sin 2n\alpha \qquad \ldots(2)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+4)\frac{\cos(s+2)\alpha}{\sin(s+2)\frac{\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds=\sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi}(2n+2)\times\left(\frac{t}{\rho}\right)^{2n-2}.\cos 2n\alpha. \qquad \ldots(3)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+2)\frac{\cos s\alpha}{\sin\frac{s\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds=\sum_{n=0}^{\infty}\frac{2(-1)^n}{\pi}(2n+2)\times\left(\frac{t}{\rho}\right)^{2n}.\cos 2n\alpha. \qquad \ldots(4)$$

And for $t > \rho$ we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+2)\frac{\sin(s+2)\alpha}{\sin(s+2)\frac{\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds = \sum_{n=1}^{\infty}\frac{2(-1)^{n}}{\pi}(2n) \times\left(\frac{\rho}{t}\right)^{2n+2}\cdot\sin 2n\alpha \qquad \ldots(5)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+2)\frac{\sin s\alpha}{\sin\frac{s\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds = \sum_{n=1}^{\infty}\frac{2(-1)^{n}}{\pi}(2n-2) \times\left(\frac{\rho}{t}\right)^{2n}\cdot\sin 2n\alpha \qquad \ldots(6)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+4)\frac{\cos(s+2)\alpha}{\sin(s+2)\frac{\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds = -\sum_{n=0}^{\infty}\frac{2(-1)^{n}}{\pi}(2n-2) \times\left(\frac{\rho}{t}\right)^{2n+2}\cdot\cos 2n\alpha \qquad \ldots(7)$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(s+2)\frac{\cos s\alpha}{\sin\frac{s\pi}{2}}\left(\frac{t}{\rho}\right)^{s}ds = -\sum_{n=1}^{\infty}\frac{2(-1)^{n}}{\pi}(2n) \times\left(\frac{\rho}{t}\right)^{2n}.\cos 2n\alpha. \qquad \ldots(8)$$