ON SPECTRA OF POLAR FACTORS OF SOME CONVEXOID OPERATORS

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Recently Putnam has obtained relations between the spectra of a hyponormal operator and its polar factors when the latter exist. In this paper similar relations are obtained for certain convexoid operators.

§ 1. A (bounded linear) operator T on a Hilbert space H is said to have a polar factorization if $T = UP$ where U is unitary and P is non-negative self-adjoint operator. If $T = UP$ is a polar factorization then $P = (T^*T)^{1/2}$ and $(TT^*)^{1/2} = U(T^*T)^{1/2}U^*$. In general the unitary factor is not unique. In case T is non-singular the polar factorization of T exists and is unique.

Recently, Putnam (1974) has investigated relations between the spectra of a hyponormal operator T and its polar factors. In this paper we obtain similar relations for certain convexoid operators in general.

§ 2. We write $B(H)$ for the Banach algebra of all operators on H. Let $w_\rho(T), \rho > 0$, denote the operator radius of T defined by

$$w_\rho(T) = \inf\{\alpha : \alpha > 0 \quad \text{and} \quad \alpha^{-1} T \in C_\rho\}$$

where C_ρ is the class of all operators having unitary ρ-dilation (Holbrook 1968 Nagy and Foias 1972). We define T to be an operator of class M_ρ $(\rho \geqslant 1)$ if

$$w_\rho[(T - zI)^{-1}] \leqslant \frac{1}{\text{dist}(z, \sigma(T))}, \quad z \notin \sigma(T)$$

where $\sigma(T)$ denotes the spectrum of T (Patel 1973, Patel and Gupta 1975, Gupta 1976). Since $w_1(T) = \|T\|$. M_1 consists of all operators satisfying condition G_1 (Stampfli 1965). Also $M_\rho \subset M_{\rho'}$ for $\rho < \rho'$. It can be shown that each class M_ρ is contained in the class of convexoid operators, that is, if $T \in M_\rho$, then convex hull of $\sigma(T)$ is $\overline{W(T)}$ the closure of the numerical range of T (Patel 1973). We use here the faithful *-representation $T \to T^0$ of $B(H)$ into $B(H^0)$ given by Berberian (1962).

Theorem 1—Let $T \in M_\rho$ and $Z_0 \in Bdry\sigma(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors in H^0 such that $(T^0 - z_0)x_n \to 0$ and $(T^{0*} - \bar{z}_0) \to 0$.

To prove this we use the following result of Patel and Gupta (1975).

Lemma 1—Let T be of class M_p and $z_0 \in Bdry\,\sigma(T)$. Let there exist a sequence $D_n = \{z: |z - z_n| < r_n\}$ of disks contained in the resolvent set of T such that $z_n \to z_0$ and $r_n^{-1}\,|z_n - z_0| \to 1$. Then z_0 is an approximate normal eigenvalue of T.

PROOF OF THEOREM 1: By changing the Hilbert space suitably (Berberian 1962, Patel and Gupta 1975) we can obtain that every approximate eigenvalue is an eigenvalue of T^0. For $n = 1, 2, \cdots$, let $S_n = \{z: |z - z_0| < 1/n\}$. Then S_n contains a point u_n of the resolvent set of T such that $|u_n - z_0| < 1/2n$. Clearly dist $(u_n, \sigma(T^0))$ is assumed in S_n: say $z_n \in \sigma(T^0)$ with dist $(u_n, \sigma(T^0)) = |u_n - z_n|$. Thus z_n is a semi-bare point of $\sigma(T^0)$ and $z_n \to z_0$. Now for each fixed n, choose a sequence $\{\lambda_k\}$ on the line segment $\widetilde{u_n z_n}$ such that

$$\frac{1}{\sqrt{k+1}} < |\lambda_k - z_n| < \frac{1}{\sqrt{k}}.$$

Then each $D_k = \{z: |z - \lambda_k| < r_k\}$ with $r_k = |\lambda_k - z_n| - 1/(k+1)$ is contained in the resolvent set of T. $\lambda_k \to z_n$ and $r_k^{-1}\,|\lambda_k - z_n| \to 1$ as $k \to \infty$. Since $T \in M_p$ implies $T^0 \in M_p$ for each n, Lemma 1 yields a unit vector x_n such that $(T^0 - z_n)\,x_n = 0$ and $(T^{0*} - \bar{z}_n)\,x_n = 0$. So we have

$$\|(T^0 - z_0)\,x_n\| = |z_n - z_0| \to 0.$$

Similarly $\|(T^{0*} - \bar{z}_0)\,x_n\| \to 0$. This completes the proof.

In what follows, for a set M of complex numbers we write $|M| = \{|z| : z \in M\}$ and $C_r = \{z: |z| = r\}$.

Theorem 2—Let T be of class M_p and $0 \notin \text{Int}\,\overline{W(T)}$. Then $|\sigma(T)| \subset \sigma(T^*T)^{1/2} \cap \sigma(TT^*)^{1/2}$.

PROOF: Let $z \in \sigma(T)$. If $z \neq 0$ then $C_{|z|}$ clearly exits $\sigma(T)$ at some point u so that $u \in Bdry\,\sigma(T)$. By Theorem 1 there exists a sequence $\{x_n\}$ of unit vectors such that $\|(T^0 - u)\,x_n\| \to 0$, $\|(T^{0*} - \bar{u})\,x_n\| \to 0$ and so $\|\big((T^{0*}T^0)^{1/2} - |u|\big)\,x_n\| \to 0$ and $\|\big((T^0T^{0*})^{1/2} - |u|\big)\,x_n\| \to 0$. Therefore it follows that $|z| \in \sigma(T^{0*}T^0)^{1/2} \cap \sigma(T^0T^{0*})^{1/2} = \sigma(T^*T)^{1/2} \cap \sigma(TT^*)^{1/2}$.

If $z = 0$ then z is a boundary point of $\sigma(T)$ and again the result follows by Theorem 1.

The following theorem has been proved by Putnam (1974) for hyponormal operators and our proof here is on the same lines.

Theorem 3—Let T be of class M_p with polar factorization $T = UP$. Suppose $re^{i\theta} \in \sigma(T)$, $r > 0$. Then $e^{i\theta} \in \sigma(U)$.

PROOF: Let $z_1 = r_1 e^{i\theta}$ where $r_1 = \max\{|z| : z = |z|\,e^{i\theta} \in \sigma(T)\}$. Then $z_1 \in Bdry\,\sigma(T)$. Therefore by Theorem 1 there exists a sequence $\{x_n\}$ of unit vectors such that $(T - z_1)\,x_n \to 0$ and $(T^{0*} - \bar{z}_1)\,x_n \to 0$ so that

$((T^{0*}T^0)^{1/2} - r_1)\ x_n \to 0$. But $(T^0 - z_1)\ x_n = U^0\ (T^{0*}T^0)^{1/2} x_n - z_1 x_n \to 0$ and $r_1 > 0$, hence $U^0 x_n - e^{i\theta} x_n \to 0$. Thus $e^{i\theta} \in \sigma(U^0) = \sigma(U)$.

Theorem 4—If $T = UP$ is convexoid, $0 \notin \overline{W(T)}$ and $\sigma(T)$ is connected and if $e^{i\theta} \in \sigma(U)$ then there exists $r > 0$ such that $re^{i\theta} \in \sigma(T)$.

PROOF: Since $0 \notin W(T)$, U is cramped, $\overline{W(T)}$ and $\overline{W(U)}$ have the same cramped sector. If

$\sigma(U) \subset \{e^{i\theta}: \theta_1 \leq \theta \leq \theta_2, \theta_2 - \theta_1 < \pi\}$ then lines $\theta = \theta_1$ and $\theta = \theta_2$ intersect $\sigma(T)$ (*see* Furato and Nakamoto 1970). Since $\sigma(T)$ is connected every ray in the sector through the points of $\sigma(U)$ intersects $\sigma(T)$ and the result follows.

§ 3. In this section we consider operators T of class R defined by the condition

$$\| T - z)^{-1} \| = \frac{1}{\text{dist}\left(z, \overline{W(T)}\right)}$$

for all $z \notin \overline{W(T)}$. Luecke (1972) has shown that $T \in R$ if and only if $Bdry W(T) \subset \sigma(T)$ so that every $T \in R$ is convexoid. We shall write $essp\,(T)$ for the essential spectrum of T defined by $essp\ (T) = \sigma(\hat{T})$ where $\hat{T}$ is the canonical image of T in the Calkin's algebra. We recall that $z \in essp_1\,(T)$, the left essential spectrum of T if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly and $(T - z)\, x_n \to 0$ (Fillmore *et al*. 1972).

Theorem 5—If $T \in R$ and $0 \notin \text{Int}\ \overline{W(T)}$ then $|\sigma(T)| \subset essp\ (T^*T)^{1/2} \cap essp\ (TT^*)^{1/2}$.

PROOF: Let $z \in \sigma(T)$. Since $T \in R$, $Bdry\ W(T) \subset \sigma(T)$ and so $Bdry W(T) \cap \pi_{00}(T) = \phi$, where $\pi_{00}(T)$ is the set of isolated eigenvalues of T of finite multiplicity. In view of the relation $Bdry\, \sigma(T) \subset essp_1\,(T) \cup \pi_{00}(T)$ (Fillmore *et al.* 1972) we have $Bdry\ W(T) \subset essp_1\ (T)$. Now let $C_{|z|}$ exit $\overline{W(T)}$ at u then $u \in essp_1\,(T)$ and so there exists a sequence $\{x_n\}$ of unit vectors, converging weakly to 0 such that $Tx_n - ux_n \to 0$ and $T^*x_n - \bar{u}x_n \to 0$. Therefore $(T^*T)^{1/2} x_n - |u|\, x_n \to 0$ and $(TT^*)^{1/2}\, x_n - |u|\, x_n \to 0$ and since $|u| = |z|$, the result follows.

Theorem 6. Let $T \in R$ and $0 \notin \overline{W(T)}$ so that T has unique polar factorization $T = UP$. Suppose $a \in \sigma(P)$ and $a \leq w(T)$, the numerical radius of T. Then $a \in |\sigma(T)|$.

PROOF: Let α be the left extreme point of $\sigma(P)$. First we show that $\alpha \in |\sigma(T)|$. Suppose not. Then $C_\alpha \cap \sigma(T) = \phi$ and since $Bdry\ W(T) \subset \sigma(T)$, $C_\alpha \subset \overline{W(T)} = \phi$. From this it follows that $\overline{W(T)}$ lies outside C_α. In fact, if $\lambda \in Bdry\ W\ T)$ then exists a sequence of unit vectors $\{x_n\}$ such that $(T - \lambda)\ x_n \to 0$ and $(T^* - \bar{\lambda})\ x_n \to 0$ so that $|\lambda| \in \sigma(P)$ and hence $|\lambda| \geq \alpha$. Therefore there exists $\varepsilon > 0$ such that $b \geq \alpha + \varepsilon$ for every $b \in |W(T)|$. Now $a \in \sigma(P) \subset \overline{W(T)}/W(U)$

(Williams 1967) implies that $\alpha \geq \gamma$, for some $\gamma \in |\overline{W(T)}|$. Thus $\alpha \geq \alpha + \varepsilon$, a contradiction. This proves that $\alpha \in |\sigma(T)|$. Next, let $a \in (P)$ and $a \leq w(T)$. If $a \notin |\sigma(T)|$, then $C_a \cap \overline{W(T)} = \phi$, and so $\alpha \leq a$ implies that $\overline{W(T)}$ lies inside c_a, which contradicts $a \leq w(T)$.

Theorem 7—Let T be of class R with polar factorization $T = UP$. If $re^{i\theta} \in \sigma(T)$, $r > 0$ then $e^{i\theta} \in \sigma(U)$.

PROOF: Let $r_1 = \max\{|z| : z = |z| e^{i\theta} \in \overline{W(T)}\}$. Since $Bdry\ W(T) \subset \sigma(T)$, $z_1 = r_1 e^{i\theta} \in Bdry\ W(T) \cap \sigma(T)$. Therefore z_1 is an approximate normal eigenvalue of T and the proof can be completed as in Theorem 3.

Theorem 8—Let T be of class R and $\theta \notin \overline{W(T)}$. Suppose $T = UP$ is the polar factorization of T. Then if $e^{i\theta} \in \sigma(U)$, there exists $r > 0$ such that $re^{i\theta} \in \sigma(T)$.

PROOF: Since T is convexoid and $\theta \notin \overline{W(T)}$, U is cramped and $\overline{W(T)}$ and $\overline{W(U)}$ have the same cramped sectors. Consequently, every ray through the points of $\sigma(U)$ intersects $Bdry\ W(T)$ and hence $\sigma(T)$. This completes the proof.

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