ON AN AFFINE CONNEXION IN ALMOST CONTACT MANIFOLD

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In this paper we have defined a connexion D satisfying some certain properties in an almost contact manifold. On the basis of these properties we have obtained some relations which are useful to establish the expressions for Nijenhuis tensors. Further a decomposition of curvature tensor has been studied. At last we have studied projective curvature tensor in the light of this decomposition.

1. Introduction

Let us consider an n-dimensional real differentiable manifold V_n of class C^∞. Let there exist in V_n a C^∞ vector valued linear function F, a C^∞ vector field T and a C^∞ 1-form A satisfying

$$F(F(X)) = -X + A(X)\,T. \quad \ldots \quad (1{\cdot}1)$$

which implies

$$\left.\begin{array}{l}\text{rank }(F) = n-1,\ F(T) = 0,\ A(F(X)) = 0\\ A(T) = 1 \text{ and } n \text{ is odd} = 2m+1 \text{ (say)}\end{array}\right\} \quad \ldots \quad (1{\cdot}1a)$$

for arbitrary vector field X. Then V_n is called an almost contact manifold and the structure (F, T, A) is called an almost contact structure (1·1a) follow from (1·1) for (Mishra 1972*b*) in consequent of (1·1)

$$F(X) = 0 \Rightarrow F(FX) = 0 \Rightarrow X{=}A(X)\,T \Rightarrow \text{there exists}$$

a unique solution $X = T$ (upto a factor) of $FX{=}0$. Hence we have (1·1*a*) (Mishra 1972*a*).

Pre-multiplication of F to (1·1) and using (1·1*a*) (Mishra 1972*b*) we get

$$F(F(FX))+F\,X = 0.$$

Pre-multiplication of F to X in (1·1), we get

$$F(F(FX))+FX{=}A(FX)\,T.$$

Using these two, we have (1·1*a*). Now (1·1*a*) implies

$$F(FT) = 0 \Rightarrow T{=}A(T)\,T \Rightarrow (1{\cdot}1a) \text{ (Blair 1971)}.$$

Let a be an eigen value of F, the corresponding eigen vector being Q. Then

$$FQ=aQ,\ F(FQ)=aF(Q)=a^2\ Q.$$

Using (1·1) in the last equation, we have

$$(a^2+1)Q=A(Q)T.$$

Two cases arise:

(a) $Q=FT$. Then $a=0$

(b) $Q\neq FT$. Then $a=\pm i$.

Thus 0 is an eigen value, the corresponding eignvector being T. The other eigenvalue $(i,-i)$ occur in pairs, the corresponding complex conjugate eigen vectors (Q, FQ) being such that $A(Q)=A(FQ)=0$. Hence we have (1·1a).

2. Affine Connexion

Let us consider an affine connexion D in V_n with torsion tensor S such that

$$(D_X A)(Y)+(D_Y A)(X)=0 \qquad (2\cdot1a)$$

$$(D_X F)(Y)+(D_Y F)(X)=0 \qquad (2\cdot1b)$$

$$D_X T=0. \qquad (2\cdot1c)$$

Theorem 2·1 We have

(a) $(D_X A)(T)=0$. (b) $(D_T A)(X)=0$, (c) $A(D_T FY)=0$. (2·2)

Proof : From (1·1a), we have

$$(D_X A)(T)+A(D_X T)=0.$$

Using (2·1c), we get (2·2a). Putting T for Y in (2·1a) and using (2·2a), we get (2.2b). Differentiating (1·1a) (Yano 1965) along T and using (2.2b), we get (2·2c).

Theorem 2.2—We have

$$D_T FY=FD_T Y. \qquad (2\cdot3)$$

Proof : Putting T for X in (2.1b) and using (2.1c), we get (2·3).

Corollary (2·1)—In V_n, we get

$$A(D_{FX}\ FY+D_{FY}\ FX)=0. \qquad (2\cdot3A)$$

Proof : Operating F on X and Y in (2·1a) and using (1·1a) (Yano 1965), we get (2·3A).

Theorem 2·3. In the almost contact manifold V_n, we have

$$D_{FY}\ FX-D_X\ Y-F(D_X FX)-F(D_{FY}\ X)+X(A(Y))T=0 \qquad (2\cdot3a)$$

$$F(D_X FY)+F(D_Y FX)+D_X Y+D_Y X=A(D_X Y+D_Y X)\ T \qquad (2\cdot 3b)$$

$$F(D_{FX}FY)+F(D_{FY}FX)+D_{FX}Y+D_{FY}X=\{(FX(A(Y))+FY(A(X))\}T, \qquad (2\cdot 3c)$$

and

$$F(D_{FX}Y)+F(D_{FY}X)-D_{FX}FY-D_{FY}FX=0. \qquad (2\cdot 3d)$$

PROOF : Operating F on Y in ($2\cdot 1b$) and using ($1\cdot 1$) and ($1\cdot 1a$), we get ($2\cdot 3a$). Operating F on (2.1b) and using ($1\cdot 1$), we have ($2\cdot 3b$). Operating F on X in ($2\cdot 3a$) and using ($1\cdot 1$), we have ($2\cdot 3c$). Operation of F on ($2\cdot 3c$) gives ($2\cdot 3d$) with the help of ($1\cdot 1$) and ($2\cdot 3A$).

Theorem 2.4—In almost contact manifold V_n, we have the following relations :

$$FS(T, X) + F[T, X] + S(FX, T) + [FX, T] = 0 \qquad (2\cdot 4)$$

$$S(T, X)+[T, X]+FS(T, FX)+[T, FX] = A(D_T X)\ T. \qquad (2\cdot 5)$$

PROOF : Torsion tensor S of the connexion D is given by

$$S(X, Y)=D_X Y-D_Y X-[X, Y].$$

Using ($2\cdot 1c$) in this, we get

$$S(T, X) = D_T X + [X, T] \qquad (2\cdot 6)$$

and

$$S(T, FX)=D_T\ FX-[T, FX]. \qquad (2\cdot 7)$$

Using ($2\cdot 6$), ($2\cdot 7$), ($2\cdot 3$) and the fact that the torsion tensor S is skew-symmetric, we get ($2\cdot 4$). Operating F on ($2\cdot 4$) and using ($1\cdot 1$) and ($2\cdot 1c$), we get ($2\cdot 5$).

Theorem 2·5—Let

$$A'(X, Y) \overset{def}{=} (D_X A)(Y) \qquad (2\cdot 8)$$

then A is skew-symmetric and

$$A'(FX, FY)+A'(X, Y) = 0. \qquad (2\cdot 9)$$

PROOF : From ($2\cdot 1a$) it follows that A' is skew-symmetric in X and Y. Now ($2\cdot 8$) gives

$$A'(FX, FY)=(D_{FX}A)(FY)$$

or

$$A'(FX, FY)=FX(A(FY))-A(D_{FX}FY).$$

Using ($1\cdot 1a$) (Yano 1965) and ($2\cdot 3a$) in this equation, we get

$$A(FX, FY)= -A(D_Y X+FD_Y FX+FD_{FX}Y-Y(A(X))T)$$

which by virtue of (1·1*a*) gives

$$A'(FX, FY) = -A(D_Y X)+Y(A(X))$$
$$= (D_Y A)(X) = A'(Y, X)$$

i.e. $$A'(FX, FY)+A'(X, Y) = 0.$$

Corollary 2·2—we have

$$A'(T, X) = 0 \quad \ldots \quad (2\cdot10)$$

$$A'(X, FY) = A'(FX, Y). \quad \ldots \quad (2\cdot11)$$

PROOF : From (2·8) and (2·2), we get (2·10). Operating F on X in (2·9), we have

$$A'(F^2X, FY) = -A'(FX, Y)$$

or $$A'(FX, Y) = A'(X, FY) - A(X)A'(T, FY).$$

using (2·10) in this equation, we have (2·11).

Theorem 2.6—Nijenhuis tensor N is given by (Mishra 1972*a*)

$$N(X, Y) \stackrel{def}{=} D_{FX}FY - D_{FY}FX - D_X Y + D_Y X + A(D_X Y - D_Y X)T$$
$$-F(D_X FY - D_{FY}X + D_{FX}Y - D_Y FX) \quad \ldots \quad (2\cdot12)$$

Then we have

$$N(X,Y) = 2(A'(X, Y)\ T + 2F((D_Y F)(X))), \quad \ldots \quad (2\cdot13)$$

PROOF : Using (2·3*a*) in (2·12) we have

$$N(X,Y) = 2D_Y X - 2D_X Y + 2FD_Y FX - 2FD_X FY - 2A(D_X Y - D_Y X)T$$
$$+ ((D_X A)(Y) - (D_Y A)(X)\}\ T.$$

Using (2·8) and the fact that

$$F((D_Y F)X - D_X F)Y) = D_Y X - D_X Y + F(D_Y FX - D_X FY)$$
$$+ A(D_X Y - D_Y X)\ T$$

and (2·1*b*) in the above expression for N, we get (2·13).

Corollary 2·3—Putting T for Y in (2·13) and using (2·10), (2·1*b*) and (2·1*c*), we get

$$N(X, T) = 0.$$

Corresponding to the Nijenhuis tensors of an almost complex manifold, there are three other tensors in V_n (Mishra 1972*a*) P which is scalar value bilinear function, Q which is vector valued, R-which is 1-form, given by

$$P(X,Y) \stackrel{def}{=} (D_Y A)(FX) - (D_{FX}A)(Y) + (D_{FY}A)(X) - (D_X A)(FY),$$

$$Q(X) \stackrel{def}{=} [T, FX]+F[X, T]+S(T, FX)+FS(X, T)$$

and $$R(X) \stackrel{def}{=} (D_X A)(T)-(D_T A)(X).$$

Theorem 2.7—For an almost contact manifold V_n, we have

$$P(X, Y) = 4A'(Y, FX) \quad \text{(2·14)}$$

$$Q(X) = 0 \quad \text{(2·15)}$$

and

$$R(X) = 0. \quad \text{(2·16)}$$

PROOF : Using the definitions of P and A', we get (2·14). Using the relation (2·4) in the definition of $Q(X)$, we get (2·15). Also by virtue of (2·2*b*) and (2·2*a*) we have (2·16).

Thus we see that in V_n, Q and R are identically vanishing. Let us suppose that $N = 0$ identically. Then (2·13) gives

$$2(A'(X, Y)T+2F((D_y F)(X)) = 0.$$

Transvecting this by A, we get

$$2A'(X, Y) = 0.$$

Consequently $P=0$. Hence we have :

Theorem 2.8—In an almost contact manifold V_n, the vanishing of Nijenhuis tensor N means vanishing of P, Q and R.

However, we see that the converse of the above Theorem is not true, in general. But. if we impose the condition that D satisfies

$$(D_X F)(Y)=0,$$

then the converse is also true.

3. Curvature Tensor

The curvature tensor K in V_n is given by

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z \quad \text{(3·1)}$$

and corresponding Ricci curvature is given by

$$(C_1^1 K)(Y, Z) \stackrel{def}{=} \text{Ric}(Y, Z)$$

where C_1^1 denotes contraction in first slot.

If B be 1- form then

$$(D_X D_Y B - D_Y D_X B - D_{[X,Y]} B)(Z) = -B(K(X,Y,Z)). \quad \text{(3·2)}$$

Theorem 3.1—In an almost contact manifold V_n as defined above, we always have

$$K(X, Y, T) = 0 \text{ and hence Ric } (Y, T) = 0. \quad (3{\cdot}3)$$

PROOF : Putting T for Z in (3.1) and using (2·1*c*), we get the result.

Theorem 3.2—In an almost contact manifold, we have

$$A'(S(X, Y), Z) + (D_X A')\ (Y, Z) - (D_Y A')\ (X, Z) + A(K(X, Y, Z)) = 0. \quad (3.4)$$

PROOF : Taking covariant derivative of (2·8), we have

$$(D_X D_Y A)\ (Z) = A'\ (D_X Y, Z) + (D_X A')\ (Y, Z). \quad (3{\cdot}4a)$$

Interchanging X and Y in (3·4*a*), we have

$$(D_Y D_X A)\ (Z) = A'\ (D_Y X, Z) + (D_Y A')\ (X, Z) \quad (3{\cdot}4b)$$

and puting $[X, Y]$ for X and Z for Y in (2·8), we get

$$(D_{[X,Y]} A)\ (Z) = A'\ ([X, Y], Z). \quad (3{\cdot}4c)$$

Subtracting the sum of (3·4*b*) and (3·4*c*) from (3·4*a*) and using (3·2), we get (3·4)

Theorem 3.3—In an almost contact manifold V_n, A′ satisfies

$$A'(S(X, Y), Z) + A'(S(Y, Z), X) + (A'(S(Z, X), Y) + A(\ (D_X S)\ (Y, Z) + (D_Y S)\ (Z, X) + (D_Z S)\ (X, Y)\ ) + A\ (S(S(X, Y), Z) + S)S(Y, Z), X) + S(Z, X), Y) + 2\ (\ (D_X A')\ (Y, Z) + (D_Y A')(Z, X) + (D_Z A')\ (X, Y)\ ) = 0 \quad (3{\cdot}5)$$

PROOF : Taking cyclic permutation of X, Y, Z in (3·4) and adding all the three equations, we get

$$A'(S(X, Y), Z) + A'(S(Y, Z), X) + A'\ (S(Z, X), Y) + A\ (K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y)\ ) + 2\ (\ (D_X A')\ (Y, Z) + (D_Y A')\ (Z, X) + (D_Z A')\ (X, Y) = 0. \quad (3{\cdot}5a)$$

Using Bianchi's first identity in (3·5*a*), we get (3·5).

Corollary 3.1—If D be symmetric connexion then we have

$$(D_X A'\ (Y, Z) + (D_Y A')\ (Z, X) + (D_Z A'\ (X, Y) = 0. \quad (3{\cdot}6)$$

PROOF : If D is symmetric connexion then

$$S(X, Y) = 0.$$

Hence (3·5) reduces to (3·6).

Keeping the idea of eqn. (3·3), let us decompose the curvature tensor K by

$$K(X, Y, Z) = \lambda(X, Y)\, FZ \qquad (3{\cdot}7)$$

where λ is a 2-form.

Theorem 3.4—We have

$$\lambda(X, Y) + \lambda(Y, X) = 0 \qquad (3{\cdot}8a)$$

$$\lambda(FX, Y) + \lambda(X, FY) = 0 \qquad (3{\cdot}8b)$$

and

$$\lambda(X, T) = 0 \qquad (3{\cdot}8c)$$

$$\lambda(FX, FY) = \lambda(X, Y). \qquad (3{\cdot}8d)$$

Proof : (3·8*a*) follows from the fact that the curvature tensor is skew-symmetric in first two slots. Now, contracting curvature tensor in first slot, we get

$$\text{Ric}\,(Y, Z) = \lambda(FZ, Y),$$

which gives (3·8*b*) by virture of the fact that Ric is symmetric in two slots. Operating F on Y in (3·8*b*) and using (1·1), we have

$$\lambda(FX, FY) - \lambda(X, Y) = A(Y)\, \lambda(T, X). \qquad (3{\cdot}8e)$$

Again operating F on Y in (3·8*e*), we have

$$-\lambda(FX, Y) + A(Y)\, \lambda(FX, T) - \lambda(X, FY) = 0.$$

Using (3·8*b*) in this, we get

$$A(Y)\, \lambda(FX, T) = 0.$$

Putting T for Y and operating F on X in this, we get (3·8*c*). By virtue of (3·8*c*), we get (3·8*d*) from (3·8*e*).

Theorem 3.5—Under the decomposition (3·7) of the curvature tensor we have

$$A(K(X, Y, Z))\,) = 0 \qquad (3{\cdot}9)$$

$$K(FX, Y, Z) + K(X, FY, Z) = 0 \qquad (3{\cdot}10)$$

$$K(T, X, Y) = 0 \qquad (3{\cdot}11)$$

and

or

$$\left.\begin{aligned} K(FX, FY, Z) &= K(X, Y, Z) \\ K(FX, FY, FY) &= K(X, Y, FZ). \end{aligned}\right\} \qquad (3{\cdot}12)$$

Proof : Using (1·1*a*) (Yano 1965) and (3·7), we have (3·9). Operationg F on X in (3·7), we have

$$K(FX, Y, Z) = \lambda(FX, Y)\, FZ.$$

Using (3·8*b*) in this equation, we get (2·10). Also

$$K(T, X, Y) = \lambda\,(T, X)\, FY$$

gives (3·11) by virtue of (3·8c). Operating F on X in (3·10), we have

$$-K(X, Y, Z)+A(X)\,K(T, Y, Z)+K(FX, FY, Z)=0.$$

Using (3·11) in this we get (3·12).

Thus under the decomposition (3·7) we see from (3·3) and (3·11) that the manifold V_n is flat in the direction T. Hence in the following discussion of curvature tensor $K(X, Y, Z)$, we exclude the possibility of any vector X, Y, Z being T.

Theorem 3.6—Under the decomposition (3·7), we have

$$\text{Ric}\,(FX, Y) + \text{Ric}\,(X, FY) = 0 \quad \ldots \quad (3\cdot13)$$

$$\text{Ric}\,(FX, FY) = \text{Ric}\,(X, Y). \quad \ldots \quad (3\cdot14)$$

PROOF : We know that

$$\text{Ric}\,(X, Y) = \lambda\,(FY, X).$$

Hence

$$\begin{aligned}\text{Ric}\,(X, FY) &= \lambda\,(F^2Y, X) = \lambda(X, Y) = \lambda(FX, FY)\\ &= -\,\lambda(FY, FX)\\ &= -\,\text{Ric}\,(FX, Y)\end{aligned}$$

which is (2·13). Now, operating F on X in (2·13), we get

$$\text{Ric}\,(F^2X, Y)+\text{Ric}\,(FX, FY) = 0.$$

Using (1·1) and (3·3) in the above equation, we have (3·14).

Theorem 3.7—An almost contact manifold V_n with $(D_XF) = 0$ is recurrent if and only if λ is recurrent with the same recurrence parameter.

PROOF : Taking covariant derivative of (3·7), we have

$$(D_UK)\,(X, Y, Z) = (D_U\lambda)(X, Y)\,FZ + \lambda(X, Y)\,(D_UF)(Z)$$

which gives

$$(D_UK)\,(X, Y, Z) = (D_U\lambda)\,(X, Y)\,FZ. \quad \ldots \quad (3\cdot15)$$

Let us suppose that V_n is recurrent then

$$(D_UK)\,(X, Y, Z) = C(U)\,\lambda(X, Y)\,FZ,$$

where $C(U)$ is recurrence parameter, From (2·15), we get

$$(\,(D_U\lambda)\,(X, Y) - C(U)\,\lambda(X, Y)\,)\,FZ = 0,$$

which implies that λ is recurrent for the same recurrence parameter.

Conversely, if λ is recurrent then from (3·15), we get,

$$(D_UK)\,(X, Y, Z) = C(U)\,\lambda(X, Y)\,FZ$$

which gives V_n as recurrent manifold.

The manifold V_n is said to be λ-symmetric iff

$$(D_U \lambda)\,(X, Y) = 0.$$

Hence from (3·15), we have :

Theorem 3.8—An almost contact manifold V_n with $(D_X F)Y = 0$ is symmetric iff it is λ-symmetric.

Corollary 3.2—Under the decomposition (3·7), we have

$$(D_X A')\,(Y, Z) + (D_Y A')\,(Z, X) + (D_Z A')\,(X, Y)$$
$$= \tfrac{1}{2}\,(A'(Z, S\,(X, Y)\,) + A'\,(X, S\,(Y, Z)\,(\,+ A'\,(Y, S\,(Z, X)\,)\,) \quad .. \quad (3\cdot16)$$

PROOF : Using (3·9) in (3·5*a*), we have (3·16).

4. Projective Curvature Tensor

The projective curvature tensor P defined by

$$P(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-1}\,\{\text{Ric}\,(Y, Z)\,X - \text{Ric}\,(X, Z)Y\} \quad (4\cdot1)$$

can be expressed in terms of λ as

$$P(X, Y, Z) = \lambda(X, Y)\,FZ - 1/n-1\,\{\lambda(FZ, Y)X - \lambda\,(FZ, X)\,\dot{Y}\} \qquad (4\cdot2)$$

Theorem 4·1—We have

(a) $(C_1^1\,P)\,(X, Y, Z) = 0$

(b) $(C_3^1\,P)\,(X, Y, Z) = 0$

(c) $P(X, Y, T) = 0$

(d) $P(T, X, Y) = -\lambda\,(FY, X)T/n-1 = -\text{Ric}(X, Y)T/n-1$

where C_1^1, C_3^1 denote contraction in first and third slot respectively.

PROOF : From (4·2), we have

$$(C_1^1\,P)\,(X, Y, Z) = \lambda(FZ, Y) - \{n\lambda(FZ, Y) - (FZ, Y)\}\,/n-1 = 0$$

and

$$(C_3^1\,P)\,(X, Y, Z) = 0 - \{\lambda(FX, Y) + \lambda(X, FY)\}\,/n - 1$$

which vanishes by virtue of (3·8*b*). Putting T for Z in (4·2), we get (4·3*c*).

Putting T for X in (4·2), we get

$$P(T, Y, Z) = \lambda(T, Y)FZ - \{T\lambda(FZ, Y) - \lambda(FZ, T)Y\}\,/n-1$$

which by virtue of (3·8*c*) gives (4·3*d*).

Theorem 4·2—We have

$$P(FX, FY, Z) = K(X, Y, Z) + \{K(Y, Z, X) + K(Z, X, Y)\} /n-1 \quad (4·4)$$

$$P(FX, FY, FZ) = P(FX, Y, Z) + P(X, FY, Z) + P(X, Y, FZ) \quad (4.5)$$

and

$$P(X, Y, Z) = P(FX, FY, Z) + P(X, FY, FZ) + P(FX, Y, FZ). \quad (4·6)$$

PROOF : (4·2) can also be written as

$$P(X, Y, Z) = K(X, Y, Z) - \{\lambda(FZ, Y)X - \lambda(FZ, X)Y\} /n-1. \quad (4·7)$$

Multiplying F to X and Y in (4·7), we get

$$P(FX, FY, Z) = K(FX, FY, Z) - \{\lambda(FZ, FY)FX - \lambda(FZ, FX)FY\} /n-1.$$

Using (3·8*d*) and (3·12) in this equation, we have

$$P(FX, FY, Z) = K(X, Y, Z) - \{\lambda(Z, Y)FX - \lambda(Z, X) FY\} /n-1$$

which by virtue of (3·7) gives

$$P(FX, FY, Z) = K(X, Y, Z) - \{K(Z, Y, X) - K(Z, X, Y) /n-1$$

which gives (4·4).

Now from (4·7), we have

(a) $P(FX, Y, Z) = K(FX, Y, Z) - \{\lambda(FZ, Y) FX - \lambda(FZ, FX)Y\} /n-1$

(b) $P(X, FY, Z) = K(X, FY, Z) - \{\lambda(FZ, FY) X - \lambda(FZ, X)FY\} /n-1$

(c) $P(FX, FY, FZ) = K(X, Y, FZ) - \{- \lambda(Z, Y)X + \lambda(Z, X)Y\} /n-1$

and

$$P(FX, FY, FZ) = K(FX, FY, FZ) - \{\lambda(Z, FY)FX + \lambda(Z, FX) FY\} /n-1$$

which by virtue of (3·8*b*) and (3·12) gives

(d) $P(FX, FY, FZ) = K(X, Y, FZ) - \{\lambda(FZ, Y,)FX - \lambda(FZ, X)FY\} /n-1.$

From (*a*), (*b*), (*c*), (*d*) and (3·10), we get (4·5). Pre-multiplication of F to X in (4·5) gives

$$P(F^2X, Y, Z) + P(FX, FY, Z) + P(FX, Y, FZ) = P(F^2X, FY, FZ),$$

or

$$\begin{aligned} P(X, Y, Z) &= P(FX, FY, Z) + P(FX, Y, FZ) + P(X, FY, FZ) \\ &\quad + A(X) \{P(T, Y, Z) - P(T, FY, FZ)\}, \\ &= P(FX, FY, Z) + P(FX, Y, FZ) + P(X, FY, FZ) \\ &\quad + A(X) \{-T\,\mathrm{Ric}\,(Y, Z) + T\,\mathrm{Ric}\,(FY, FZ)\}/n-1, \end{aligned}$$

which gives (4·6).

Taking the covariant derivative of (4·7), we have

$$(D_UP)(X, Y, Z) = (D_UK)(X, Y, Z) - \frac{1}{n-1}\{(D_U\lambda)(FZ, Y)X - (D_U\lambda)(FZ, X)Y + \lambda(D_U FZ, Y)X - \lambda(D_UFZ, X)Y - \lambda(FD_U Z, Y)X + \lambda(FD_UZ, X)Y\}. \quad \ldots \quad \ldots \quad (4\cdot8)$$

Theorem 4·3—An almost contact manifold V_n with $(D_XF) = 0$ is projectively recurrent iff λ is recurrent for the same recurrence parameter.

Proof : It is projectively recurrent for the recurrence parameter C, therefore from (4·8), we have

$$C(U)P(X,Y,Z) = (D_UK)(X,Y,Z) - \{(D_U\lambda)(FZ,Y)X - (D_U\lambda)(FZ,X)Y/n-1,$$

or

$$\{C(U)\lambda(X,Y) - (D_U\lambda)(X,Y)\}FZ - [\{C(U)\lambda(FZ,Y) - (D_U\lambda)(FZ,Y)\}X - \{C(U)\lambda(FZ,X) - (D_U\lambda)(FZ, X)\}Y]/n-1 = 0.$$

This is true for arbitrary vector fields X, Y, Z and hence

$$(D_U\lambda)(X,Y) = C(U)\lambda(X,Y).$$

Conversely if λ is recurrent with $(D_XF)Y = 0$ then

from Theorem (3·7) we have V_n to be recurrent for the same recurrence parameter. Hence from equation (4·8), we have

$$(D_UP)(X,Y,Z) = C(U)\ \{K(X,Y,Z) - \frac{1}{n-1}(\lambda(FZ,Y)X - \lambda(FZ,X)Y)\},$$

or

$$(D_UP)(X,Y,Z) = C(U)\,P(X,Y,Z).$$

Hence V_n is projectively recurrent.

Theorem 4·4—Under the decomposition (3·7) an almost contact manifold V_n can not be projectively flat.

Proof : Let us suppose that V_n is projectively flat then from (4·2), we have

$$(\lambda X,Y)FZ = \frac{1}{n-1}\{\lambda(FZ,Y)X - \lambda(FZ, X)Y\}.$$

Transvecting this by A and using (1·1a)3, we get

$$A(X)\lambda(FZ,Y) = A(Y)\lambda(FZ,X)$$

which gives

$$A(X)\lambda(Y,Z) = A(Y)\lambda(X,Z) \quad \ldots \quad \ldots \quad \ldots \quad (4\cdot9)$$

which is impossible. Hence the theorem.

From (4·9), we can have

$$\lambda(X,Y) = 0.$$

This suggests that V_n will be projectively flat if it is flat manifold which is well known.

Remark

An almost contact metric manifold whose structure tensors are Killing fields is called nearly cosymplectic manifold (Blair 1971). If (2·1*c*) holds in a nearly cosymplectic manifold then the results obtained above are true for this manifold.

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