APPLICATION OF L-OPERATOR IN THE SOLUTION OF CERTAIN INTEGRAL EQUATION

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The integral equation involving the G-function (Erdelyi 1953, p. 207) as kernel has been transformed, by introducing new Gamma function factors into the integrand by means of operator L, into another integral equation with a symmetrical Fourier kernel introduced by Fox (1961) and the solution is then immediate. L and L^{-1} denote the Laplace transform and its inverse. Later some special cases are discussed.

1. INTRODUCTION

Fox (1971, 1972) and Verma (in press) have solved certain integral equations, by utilising the powers of L and L^{-1} operators to annihilate the Gamma function factors from the integrand. Here we have employed the power of operator L of introducing new Gamma function factors into the integrand for transforming the given integral equation into another integral equation involving a symmetrical Fourier kernel and the solution is then immediate. This Fourier kernel is a generalisation of a large variety of functions that occur frequently in various branches of mathematics.

We shall develop the solution of the integral equation of the type

$$\int_0^\infty G^{q,0}_{p,2q}(xu)\, f(u)\, du = \phi(x), \ (x>0) \quad \ldots \quad (1\cdot1)$$

where ϕ is given and f is the function to be determined.

In (1·1),

$$G^{q,0}_{p,2q}(x) = (2\pi i)^{-1} \int_T M_{2q,p}(s)\, x^{-s}\, ds \quad \ldots \quad (1\cdot2)$$

where

$$M_{2q,p}(s) = \prod_1^q \Gamma(b_i + c\, s) \Big\{ \prod_1^q \Gamma(b_i + c - cs) \prod_1^p \Gamma(a_i - c + cs) \Big\}^{-1}$$
$$= \mathscr{M}\,[\,G^{q,0}_{p,2q}(x)]\,, \quad \ldots \quad (1\cdot3)$$

$\mathscr{M}$ denotes Mellin transform.

We make the following simplifying assumptions :

(i) $c > 0$, $h = c(2q - p) > 0$;

(ii) all the poles of the integrand of (1·2) are simple;

(iii) the contour T is a straight line parallel to the imaginary axis in the s $(= \sigma + i\tau)$ plane and the poles of $\Gamma(b_i + cs)$ lie to the left of it.

We now use the asymptotic expansion of the Gamma function (Whittaker and Watson 1915) :

$$\log \Gamma(s+a) = (s+a-\tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log(2\pi) + O(s^{-1}) \quad .. \quad (1\cdot4)$$

where $|\arg s| < \pi$ and 0 is the order symbol, in finding the asymptotic expansion of $M_{2q,p}(s)$, $s = \sigma + i\tau$, σ and τ real when $|\tau|$ is large. The result is

$$M_{2q,p}(s) = |\tau|^{h(\sigma-1/2)} \exp\{i\tau(h \log|\tau| - B)\} \{Q + O(|\tau|^{-1})\} \quad (1\cdot5)$$

where B is a constant and Q is a constant which may have one value when τ is large and positive and another value when τ is large and negative. From (1·5), it follows that if $\sigma < \frac{1}{2}$ then the integral of (1·2) is uniformly convergent with respect to x. It can be extended for the case $\sigma = \frac{1}{2}$.

2. The Laplace Transform and Mellin Transform

The Laplace transform of $\phi(x)$ is defined by the relation

$$L\{\phi(x)\} = \int_0^\infty e^{-xt} \phi(x)\, dx = \psi(t). \quad .. \quad .. \quad .. \quad (2\cdot1)$$

If $\mathcal{M}[h(u)] = H(s)$ and $\mathcal{M}[f(u)] = F(s)$, then the Mellin-Parseval Theorem states that

$$\int_0^\infty h(u) f(u)\, du = (2\pi i)^{-1} \int_C H(s) F(1-s)\, ds \quad .. \quad .. \quad (2\cdot2)$$

where C is a suitable contour in the s-plane and $\mathcal{M}$ denotes the Mellin transform.

3. The Solution of (1·1) as an Integral Equation

Theorem—If

(i) $c > 0$, $\mathrm{Re}(a_i) > c/2$, $i = 1, \ldots, p$;

(ii) $f(x) \in L_2(0, \infty)$;

(iii) $s^{h(s-1/2)} F(1-s) \in L(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$;

$h = c(2q - p) > 0$;

(iv) $F(1-s) \in L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$;

(v) $y^{-1/2} f(y) \in L(0,\infty)$, where $f(y)$ is of bounded variation near the point $y = x$, then the solution of (1·1), as an integral equation for $f(u)$ is

$$f(x) = \int_0^\infty G^{q,p}_{2p,\,2q}\,(x\,u)$$

$$\times \left[t^{a_1} L\left\{ \tau^{a_1-1} \left[t^{a_1} L\left\{ \ldots\, \tau^{a_{p-1}-1} \left[t^{a_p} L\left\{ u^{a_p-1}\ \phi(u^c) \right\} \right]_{t=1/\tau} \right\} \ldots \right.\right.\right.$$

$$\left.\left. \ldots \right\} \right]_{t=1/u} du. \qquad .. \quad (3\cdot1)$$

PROOF : Firstly, we apply (2·2) to the left-hand side of (1·1). For large positive u and $x>0$, the asymptotic expansion of $G^{q,\,0}_{p,\,2q}\ (xu)$ discussed as in Fox (1961) and conditions (i) and (ii), allow us to use Theorem 72 on p. 95 of Titchmarsh (1937) with $k = \frac{1}{2}$. Thus, we can apply (2·2) to the left-hand side of (1·1). Using (1·3), we obtain

$$\phi(x) = (2\,\pi\,i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2q,\,p}\,(s)\ x^{-s}\,F\,(1-s)\ ds, \quad x > 0, \qquad .. \quad (3\cdot2)$$

where $M_{2q,\,p}\,(s)x^{-s}$ and $F\,(s)$ are Mellin transforms of $G^{q,\,0}_{p,\,2q}\ (xu)$ and $f(u)$ respectively, and the contour in the s-plane is the straight line $s = \frac{1}{2}+i\tau$, τ varies from $-\infty$ to ∞.

In this section of the proof, we shall try to introduce p new Gamma function factors into the integrand of (3·2) by using the power of operator L of introducing new Gamma function factors into the integrand. In the first instance, we introduce the *p-th* new Gamma function factor $\Gamma\,(a_p-cs)$ into the integrand by using the technique of operator L. Then, in similar manner, we can introduce the remaining $(p-1)$ factors, namely, $\Gamma(a_1-cs)$, $\Gamma(a_2-cs), \ldots\ldots, \Gamma(a_{p-1}-cs)$, and thus we can arrive at the result with symmetrical Fourier kernel and the solution then would be immediate.

Now we use operator L to introduce the Gamma function factor $\Gamma\,(a_p-cs)$, $c>0$, $\mathrm{Re}(a_p\,) > c/2$, into the integrand of (3·2),

Considering $\phi(x^c)$, we are led to the result

$$\phi(x^c) = (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2q,\,p}(s)\ x^{-cs}\ F\,(1-s)\ ds. \qquad .. \quad (3\cdot3)$$

Using operator L, we find

$$L\left\{ x^{a_p-1} \phi(x^c) \right\}$$

$$= \int_0^\infty e^{-tx} x^{a_p-1} \left\{ (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2q,p}(s)\, x^{-cs} F(1-s)\, ds \right\} dx. \quad .. \quad (3\cdot4)$$

Now using (1·5), one can write for large $|\tau|$, $s = \frac{1}{2}+i\tau$,

$$M_{2q,p}(s)\, F(1-s) = s^{h(s-1/2)} F(1-s) \{Q_1 + 0(s^{-1})\}, \quad .. \quad (3\cdot5)$$

where Q_1 is a constant which may have one value when τ is large and positive and another value when τ is large and negative.

Since $s = \frac{1}{2}+i\tau$, the real power of x in (3·4) is $\mathrm{Re}(a_p) - c/2 - 1$, which by condition (i), exceeds -1. Also, by (3·5) and Condition (iii), the terms in s in (3·4) belong to $L(\frac{1}{2} - i\infty, \frac{1}{2}+i\infty)$. Hence ,the integral in (3·4) is an absolutely convergent double integral and we can integrate with respect to x. The result, thus, found is

$$L\left\{ x^{a_p-1} \phi(x^c) \right\}$$

$$= (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2q,p}(s)\, \Gamma(a_p - cs)\, t^{-a_p+cs} F(1-s)\, ds. \quad .. \quad (3\cdot6)$$

In order to apply again the operator L, we have to write $t=1/\tau$ in (3·6), thus we obtain

$$\left[t^{a_p} L\left\{ x^{a_p-1} \phi(x^c) \right\} \right]_{t=1/\tau}$$

$$= (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2q,p}(s)\, \Gamma(a_p - cs)\, \tau^{-cs} F(1-s)\, ds. \quad .. \quad (3\cdot7)$$

Now again, we apply operator L (3·7) in order to introduce the new Gamma function factor $\Gamma(a_{p-1} - cs)$ as in the previous case and justifying the change in the order of integration, we are led to the result:

$$L\left\{ \tau^{a_{p-1}-1} \left[t^{a_p} L\left\{ x^{a_p-1} \phi(x^c) \right\} \right]_{t=1/\tau} \right\}$$

$$= (2\pi i)^{-1} \int_{1/2-i\infty}^{1/2+i\infty} M_{2q,p}(s)\, \Gamma(a_{p-1} - cs)\, \Gamma(a_p - cs)\, x^{-a_{p-1}+cs} F(1-s)\, ds. \quad .. \quad (3\cdot8)$$

Hence, by means of (p—2) successions of L operator, we can arrive at the result

$$\left[t^{a_1} L\left\{\tau^{a_1-1}\left[t^{a_2} L\left\{\ldots \tau^{a_{p-1}-1}\left[t^{a_p} L\left\{x^{a_p-1}\,\phi(x^c)\right\}\right]_{t=1/\tau}\right\}\ldots\right\}\right]_{t=1/x}$$

$$=(2\pi i)^{-1}\int_{1/2-i\infty}^{1/2+i\infty} M_{2q,\,2p}(s)\; x^{-cs}\, F(1-s)\, ds, \qquad (3\cdot 9)$$

where

$$M_{2q,\,2p}(s)=\prod_1^q \Gamma(b_i+cs)\prod_1^p \Gamma(a_i-cs)\times\left\{\prod_1^q \Gamma(b_i+c-cs)\prod_1^p \Gamma(a_i-c+cs)\right\}^{-1}. \qquad (3\cdot 10)$$

If we write the left-hand side of (3·9) equal to $\psi(x)$ and replace x by $x^{1/c}$, then (3·9) takes the concise form

$$\psi(x)=(2\pi i)^{-1}\int_{1/2-i\infty}^{1/2+i\infty} M_{2q,\,2p}(s)\; x^{-s}\, F(1-s)\, ds \qquad (3\cdot 11)$$

or

$$\psi(x)=(2\pi i)^{-1}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\left[G^{\,q,\,p}_{2p,\,2q}(u)\right] x^{-s}\, F(1-s)\, ds \qquad (3\cdot 12)$$

in the light of (1·3).

On using [Fox 1965, p.391, eqn. (12)] to the right-hand side of (3·12), it can be expressed by an integral involving the product of $G^{q,\,p}_{2p,\,2q}(xu)$ and $f(u)$. The result, thus obtained is

$$\int_0^\infty G^{q,\,p}_{2p,\,2q}(xu)\, f(u)\, du=\psi(x). \qquad (3\cdot 13)$$

Since $G^{q,\,p}_{2p,\,2q}(xu)$ is a symmetrical Fourier kernel, we are thus led formally to the solution as:

$$f(x)=\int_0^\infty G^{q,\,p}_{2p,\,2q}(xu)\;\psi(u)\; du \qquad (3\cdot 14)$$

$$=\int_0^\infty G^{q,\,p}_{2p,\,2q}(xu)\times$$

$$\times\left[t^{a_1} L\left\{\tau^{a_1-1}\left[t^{a_2} L\left\{\ldots \tau^{a_{p-1}-1}\left[t^{a_p} L\left\{u^{a_p-1}\,\phi(u^c)\right\}\right]_{t=1/\tau}\right\}\ldots\right\}\right]_{t=1/u} du \qquad (3\cdot 15)$$

4. Particular Cases

(i) By taking $c=1$, $q=1$, $p=0$, $b_1=\nu$, our theorem leads to

Corollary 1—Under the conditions of the theorem, the integral equation

$$\int_0^\infty J_{2\nu}\ (2(xu)^{1/2}) f_1(u)\ du = \phi_1(x),\ (x>0) \qquad \text{(4·1)}$$

has solution

$$f_1(x) = \int_0^\infty J_{2\nu}\ (2(xu)^{1/2})\ \phi_1(u)\ du \qquad \text{(4·2)}$$

which is the classical solution of (4·1) as $J_{2\nu}(2(xu)^{1/2})$ is a symmetrical Fourier kernel, and it can be obtained directly only by interchanging $f_1(u)$ and $\phi_1(x)$ in (4·1), where $J_\nu(x)$ is Bessel function.

(ii) With $c=1$, $p=2$, $q=2$, our theorem leads to another

Corollary 2—Under the conditions of the theorem, the integral equation

$$\int_0 G_{2,4}^{2,0}\ (xu)\ f_2\ (u)\ du = \phi_2\ (x),\ (x > 0) \qquad \text{(4·3)}$$

has solution

$$f_2(x) = \int_0 G_{4,4}^{2,2}\quad (xu)$$
$$\times \left(\left[t^{a_1} L \left\{ \tau^{a_1-1} \left[t^{a_2} L \left\{ u^{a_2-1} \underset{2}{\phi}\ (u) \right\} \right]_{t=1/\tau} \right\} \right]_{t=1/u} \right) du. \qquad \text{(4·4)}$$

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