A WEAK JET OF AN INCOMPRESSIBLE PSEUDOPLASTIC FLUID

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A weak jet of an incompressible pseudoplastic fluid from an orifice of non-vanishing width under full expansion in a moving medium is considered. As the jet is weak, therefore, there is a very small difference between the velocity of the jet and that of the medium in which it emerges. In the present note we discuss the flow behaviour of such a jet. Boundary layer equations for the flow are approximated by retaining the terms only upto the order one and are solved in a closed form for $0 < n < 1$, where n is the flow behaviour index. Graphs for the velocity profiles are drawn.

Notation

J = flux of momentum across any cross section perpendicular to jet axis
U_e = velocity of the medium in which jet emerges
U_{10} = velocity just at the exit of the jet
h = width of the orifice
n = flow behaviour index of pseudoplastic fluid
u, v = velocity components in the direction of x and y respectively
x = direction along the jet axis
y = direction perpendicular to the jet axis
a = ν/U_e
ν = kinematic viscosity of the pseudoplastic fluid

Introduction

A jet issuing from an orifice and mixing with a surrounding fluid at rest is a classical problem in jet flows. This has been discussed in most of the text books on the fluid mechanics. Under the same type of the conditions the jet of the pseudo plastic fluid is studied by Kapur (1962) and Gutfinger (1964) separately.

When the jet emerges through infinitely narrow orifice, there is a sudden decrease in the area at the orifice. If this produces a very large velocity near the orifice in comparison to the velocity of a medium in which it emerges, the jet is considered to be a strong jet. Now consider the flow from the two dimensional orifice of nonvanishing width under full expansion (The flow is said to be under full expansion when the pressure of the flow at the exit is nearly equal to that of the surrounding stream). In some practical situations there is a very small difference between the velocity of the jet and that of the medium in which it emerges. Such a jet is known as a weak jet.

Here we shall study a weak jet of the pseudo plastic fluid in a medium with a uniform velocity U_e. In the down stream direction, ultimately the velocity of the jet will be very small in comparision to U_e.

Basic Equations

With the usual approximation, the boundary layer equations for the two dimensional steady motion of an incompressible pseudo plastic fluid under no external pressure are given by (Kapur 1963).

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial}{\partial y}\left[\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right], \ (0 < n < 1) \quad \ldots \quad \ldots \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \ldots \quad \ldots \quad (2)$$

The necessary boundary conditions are

$$\text{at } x = 0,\ u = U_{10} + U_e = \text{const. for } y < \left| \frac{h}{2} \right| \quad \ldots \quad \ldots \quad (3)$$

$$= U_e. \quad = \text{const. for } y > \left| \frac{h}{2} \right|$$

$$y = 0 \ v = 0 \text{ and } \frac{\partial u}{\partial y} = 0 \quad \ldots \quad \ldots \quad (4)$$

$$y \to \propto u = U_e. \quad \ldots \quad \ldots \quad (5)$$

Solution of the Equations

Since the velocity is slightly different from that of a surrounding medium, it is assumed that the velocity difference $u_1 = u - U_e$ is very small in comparison to $U_e (u_1 << U_e)$ and v_1 ($v = v_1$ is taken) is approximately of the order u_1. We then find the approximate solution of eqn. (1).

Substituting in eqn. (1), and neglecting the terms of $\frac{u_1}{U_e}$ and $\frac{v_1}{U_e}$, one obtains

$$\frac{\partial u_1}{\partial x} = a \frac{\partial}{\partial y}\left[\left| \frac{\partial u_1}{\partial y} \right|^{n-1} \frac{\partial u_1}{\partial y} \right] \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where $a = \frac{\nu}{U_e}$ . .. . (7)

Since the pressure is constant and the motion is steady, the flux of momentum across any cross section perpendicular to the jet axis is constant. Hence

$$J = 2\rho \int_0^\infty u\,(u - U_e)\,dy = 2\rho\, U_e \int_0^\infty u_1 d_y = \text{const.} \quad .. \quad (8)$$

We try to find out the similarity solutions of eqn. (6) in the form of a stream function

$$\psi_1 = aU_e\, F(\eta)\ .\ \text{where } \eta = \frac{y}{(ax)p} \quad .. \quad .. \quad (9)$$

$$a = \frac{1}{U_e}\ \frac{1}{2n}^{1/(n-1)} \quad . \quad .. \quad .. \quad (10)$$

In order to have the similarity solutions the terms on both the sides of eqn. (6) should be free from x. This consideration gives us the value of p which is equal to $\frac{1}{2n}$. Then

$$u_1 = aU_e\ F'\ .\ \frac{1}{(ax)^{1/2n}} \quad .. \quad .. \quad .. \quad (11)$$

$$J = 2\rho U_e^2 a\ .\ \int_0^\infty F'\, d\eta \quad .. \quad .. \quad .. \quad (12)$$

where the dash denotes the differentiation w.r.t. η. Substituting the values of u_1, $\frac{\partial u_1}{\partial x}$, $\frac{\partial u_1}{\partial y}$ in (6), we obtain, after a considerable simplification

$$F''\eta + F' = -n \mid F'' \mid^{n-1}\ F'''. \quad .. \quad .. \quad .. \quad (13)$$

The corresponding boundary conditions will reduce to

$$\text{at } \eta = 0,\ F = F'' = 0 \quad .. \quad .. \quad .. \quad (4a)$$

$$\eta \to \infty,\ F' = 0. \quad .. \quad .. \quad .. \quad (5a)$$

Now as y increases, u_1 decreases and hence $\frac{\partial u_1}{\partial y}$ will be a negative quantity. It amounts to be F'', a negative quantity. Then integrating (13), we get

$$(-F'')^n = n.\eta\, F' + d \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

and in view of equation (4*a*)

$$d = 0,$$

therefore

$$-F''\,(F')^{-1/n} = (n)^{1/n}\,(\eta)^{1/n} \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

In order to solve (15), we substitute

$$\zeta = c\eta \text{ and } F(\eta) = \frac{2n}{c^{1-n}}\,\phi\,(\zeta). \qquad \ldots \quad \ldots \quad (16)$$

where c is any arbitrary constant.

Now eqn. (16) reduces to

$$-\phi''(\phi')^{-1/n} = (n)^{1/n}\,\zeta^{1/n} \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

where now the dash denotes the differentiation w.r.t. ξ.
Integrating, we obtain

$$\phi'^{\,-\left(\frac{1-n}{n}\right)} = c' + B\,\zeta^{\frac{n+1}{n}} \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

where $$B = n^{1/n}\,\frac{1-n}{1+n} \quad . \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

Since we have introduced one arbitrary constant c through (16), we can select an arbitrary condition. Say it is

$$\phi'\,(0) = B^{-\left(\frac{n}{1-n}\right)} . \qquad \ldots \quad \ldots \quad \ldots \quad (20)$$

This gives

$$c' = B.$$

Hence

$$(\phi')^{-\left(\frac{1-n}{n}\right)} = B\,(1+\zeta^{\frac{n+1}{n}})$$

i.e.

$$\phi' = \beta\,[\,1+\zeta^{\frac{n+1}{n}}\,]^{\frac{-n}{1-n}} \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

where

$$\beta = \frac{1}{n^{\frac{1}{1-n}}}\left(\frac{1+n}{1-n}\right)^{\frac{u}{1-n}} . \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

With the help of eqn. (21), the velocity profiles for the flow is plotted for the different values of n (Fig. 1).

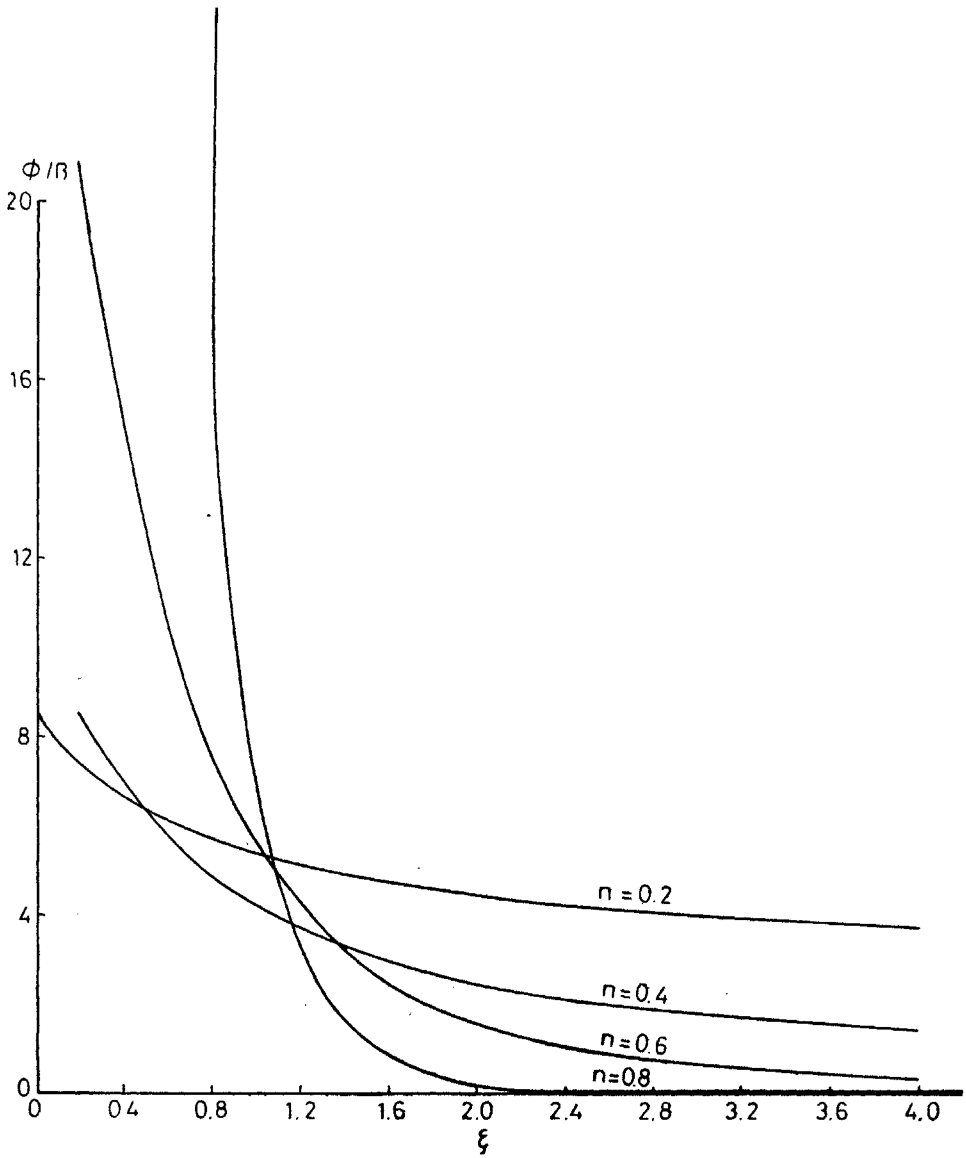

FIG. 1. Velocity profiles for the different values of n.

The free constant c can be determined from eqn. (12)

$$\frac{J}{2\rho\, U_e^2 .\, a} = \text{constant} = \int_0^\infty F'\, d\eta = c^{\frac{2n}{1-n}} \int_0^\infty \phi'\, d\zeta\,.$$

Substituting the value of ϕ' and selecting a constant $\frac{J}{2\rho\, U_e^2 .\, a}\,\frac{1}{\beta}$ to be unity, we obatin

$$1 = c^{\frac{2n}{1-n}} \int_0^\infty \left(1+\zeta^{\frac{n+1}{n}}\right)^{\frac{n}{1-n}} d\,\zeta \qquad \ldots \quad \ldots \quad \ldots \quad (23)$$

Integral on the R.H.S. is convergent for $0 < n < 1$. The following Table gives the value of c, obtained by numerical integration, for some values of n.

n	c
0·2	0·020631
0·4	0·949160
0·6	1·006905
0·8	1·053406

Discussion of the Results

The jet of the pseudoplastic fluid in a stationary medium is discussed by Kapur (1962) and Gutfinger (1964). Comparing the results for the present jet it will be observed that jet in the stationary medium attains a zero velocity much more rapidly than the jet in the moving medium attains the velocity U_e.

Figure 1 shows that as n increases $\frac{u_1}{u_{1\,max}}$ decreases more rapidly and becomes a constant after some values of ζ. This ζ is smaller in case of larger n.

The spread of the jet will be proportional to $\left(\frac{\nu}{U_e}\, x\right)^{\frac{1}{2n}}$. At a particular station the spread will increase with the increase in the value of n. For a particular value of n, the spread will also increase as the jet moves downward. This is also the case for the jet of a Newtonian fluid.

References

Gutfinger, C., and Shinnar, R. (1964). Velocity distributions in two-dimensional laminar liquid-into-liquid jets in power-law fluids. *A.I.Ch.E.Jl*, **10**, 631-39.

Kapur, J. N. (1962). On the two dimensional jet of an incompressible pseudo-plastic fluid. *J. phys. Soc. Japan*, **17**, 1303-9.

Kapur, J. N., (1963). A note on boundary layer equations for a power-law fluid. *J. phys. Soc. Japan*, **18**, 344.