ON A CLASS OF FUNCTIONS SCHLICHT IN THE UNIT DISC

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Let K_1 denote the class of functions $F(z) = z + b_2 z^2 +$ which are regular in $E = [z \mid |z| < 1]$ and satisfy the condition: $F(z) = [f(z) + zf'(z)]/2$, where $f(z) = z + a_2 z^2 + \ldots$ is regular, univalent and convex in E. It is known that K_1 is a sub-class of the class of univalent functions in E. In the present paper we determine the radius of starlikeness of the class K_1 and the radius of close-to-convexity of nth partial sum of functions belonging to this class.

Let K denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 \ldots$ which are regular, univalent and convex in $|z| < 1$ and let K_1 be the class of function

$$F(z) = \frac{1}{2}\Big[f(z) + z f'(z) \Big]. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $f(z) \in K$. Rahmanov (1951) proved that if $F(z) \in K_1$, then $F(z)$ is univalent in $|z| < 1$. Livingston (1966) showed that $F(z)$ is actually close-to-convex in $|z| < 1$ and convex in $|z| < \frac{1}{2}$. Goel (1971) found a lower bound $(= \sqrt{4/5})$ for the radius of starlikeness of the class K_1 and remarked that perhaps the exact value of the radius of star-likeness of K_1 was $\frac{2\sqrt{2}}{3}$ Goel also proved that each partial sum of $F(z) \in K_1$ was close-to-convex in $1/3$. However, his proof contains a flaw because he has implicitly assumed that each partial sum of a convex function is convex in $|z| < 1/3$ which is not correct (Szegö 1928). In the present note we determine the precise radius of starlikeness of the class K_1 and sharp radius of close-to-convexity of partial sums of functions of the class K_1. We establish these results by determining in each case the extremal function corresponding to the problem under consideration.

If $\theta(z) = \sum_{n=0}^{\infty} p_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} q_n z^n$ are any two functions, then by $\theta(z) * \psi(z)$ we shall mean the Hadamard product of $\theta(z)$ and $\psi(z)$, that is,

$$\theta(z) * \psi(z) = \sum_{n=0}^{\infty} p_n q_n z^n.$$

We shall need the following results :

Lemma 1. (Ruscheweyh and Sheil-Small (1973)—If $\phi(z)$ is convex and $g(z)$ is starlike in $|z| < 1$, then $\phi(z) * g(z)$ is starlike in $|z| < 1$.

Lemma 2. (Ruscheweyh 1972*b*)—Let $g(z)$ be regular in $|z| < 1, g(0) = 0$, $g'(0) = 1$. If there are numbers $\alpha \in (-\pi/2, \pi/2)$ and β and γ with $|\beta| \leqslant 1$, $|\gamma| \leqslant 1$, such that

$$\text{Re } [e^{i\alpha} (1-\beta z)(1-\gamma z) g'(z)] > 0, \ (|z| < 1),$$

then $f(z) * g(z)$ is close-to-convex in $|z| < 1$ for all $f(z) \in K$.

We now prove the following:

Theorem 1—If $F(z) \in K_1$ then $F(z)$ maps $|z| < \sqrt{7/8}$ onto a domain which is starlike with respect to the origin. Further, the constant $\sqrt{7/8}$ can not be replaced by any larger one.

PROOF : Let

$$F(z) = z + b_2 z^2 + b_3 z^3 + \ldots\ldots$$

be an arbitrary function of K_1 and let us suppose that $F(z)$ corresponds to the function

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

of the class K. From (1) it follows that

$$b_n = \frac{n+1}{2} a_n . \qquad \ldots \quad \ldots \quad \ldots \qquad (2)$$

Let us assume that $F(z)$ is starlike in $|z| < \rho_0$. Hence

$$\frac{1}{\rho_0} F(\rho_0 z) = z + \rho_0 z^2 + \rho_0{}^2 z^3 + \ldots$$

is starlike in $|z| < 1$.

If $$\varphi(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n ,$$

then $\phi(z) \in K$ and consequently the function

$$F_1(z) = \frac{1}{2}\left[\phi(z) + z\phi'(z)\right] = \frac{1}{2}\sum_{n=1}^{\infty}(n+1)z^{n+1} \in K_1. \qquad \ldots \qquad (3)$$

We now see (using (2)) that

$$\frac{1}{\rho_0} F(\rho_0 z) = \frac{1}{\rho_0} F_1(\rho_0 z) * f(z). \qquad \ldots \quad \ldots \quad \ldots \qquad (4)$$

Since $f(z) \in K$, in view of Lemma 1 it follows from (4) that the radius of starlikeness of the class K_1 is the circle in which $F_1(z)$, defined by (3), is starlike. The function

$F_1(z) \in K_1$ and one easily verifies that $F_1(z)$ fails to map $|z| < \rho$ onto a star-like domain if $\rho \geqslant \sqrt{7/8}$. This shows that the constant $\sqrt{7/8}$ can not be replaced by any larger one. The proof of Theorem 1 is therefore complete.

Let us denote by $K_{2\lambda}$, $0 \leqslant \lambda \leqslant 1$. the class of functions

$$g(z) = \lambda f(z) + (1-\lambda)\, z f'(z),$$

where $f(z) \in K$. We have $K_2 = K$ and $K_0 = S^*$, where S^* is the class of starlike functions in $|z| < 1$. The functions of the class $K_{2\lambda}$ are necessarily close-to-convex in $|z| < 1$. Proceeding as in Theorem 1, we obtain the following:

Theorem 2—If $g(z) \in K_{2\lambda}$, then $g(z)$ maps the disc $|z| < r_0$ onto a domain which is starlike with respect to the origin. The constant r_0, which is the smallest positive root of the equation

$$\lambda(2\lambda-1)^2 r^4 - 2\,(4(2\lambda^2-1) - 3\lambda(2\lambda-1))\,r^2 - 8 + 9\lambda = 0.$$

can not be replaced by any larger one as in shown by the function

$$g_1(z) = \lambda \frac{z}{1-z} + (1-\lambda)\frac{z}{(1-z)^2} = \sum_{n-1}^{\infty} [\,\lambda + (1-\lambda)\, n\,]\, z^n \in K_{2\lambda}. \qquad (5)$$

Theorem 3—Let $F(z) = z + \sum_{k=2}^{\infty} b_k z^k \in K_1$. Then for each n, $n \geqslant 2$, the n-th partial sum, $s_n(z) = z + \sum_{k=2}^{n} b_k z^k$, is close-to-convex (and hence univalent) in $|z| < r_n$, where r_n is the positive root of the equation

$$\phi_n(r) \equiv 2-(n+1)(n+2)r^n - 2n(n+2)r^{n+1} - n(n+1)r^{n+2} = 0.$$

For each even n, the constant r_n cannot be replaced by any larger one.

PROOF : Let

$$\sigma_n(z) = \frac{1}{2}\sum_{k=1}^{n} (k+1)\, z^k,$$

so that $\sigma_n(z)$ is the n-th partial sum of the function $F_1(z)$, defined by (3). Since $\phi_n(0) > 0$ and $\phi_n(1) < 0$, we see that $0 < r_n < 1$. Let

$$\mu_n(z) = \frac{1}{r_n}\, \sigma(r_n z).$$

Now,

$$2\mu'_n(z) = 2\,\sigma'_n(r_n z) = \frac{2}{(1-r_n z)^3} - (n+1)(n+2)\frac{r_n^n z^n}{(1-r_n z)^2} + n(n+1)\frac{r_n^{n+1} z^{n+1}}{(1-r_n z)^2} - 2r_n^{n+1}\frac{z^{n+1}}{(1-r_n z)^3} \quad .. \qquad (6)$$

and a little calculation, together with the definition of r_n, shows that in $|z| < 1$,

$$2\text{Re}\ [(1-r_n z)^2\ \mu'_n(z)]$$

$$\geqslant \frac{2}{1+r_n|z|} - (n+1)(n+2)\, r_n^n\, |z|^n - n(n+1)\, r_n^{n+1}\, |z|^{n+1}$$

$$+\ 2r_n^{n+1}\ \frac{|z|^{n+1}}{1+r_n|z|}\ 0. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

In view of (2) and Lemma 2, we, therefore, conclude that for each

$$f(z) = z + \sum_{n=2}^{n} a_n z^n \in K,$$

$$\mu_n(z) * f(z) = \frac{1}{r_n}\, \sigma_n(r_n z) * f(z) = \frac{1}{r_n}\, s_n(r_n z)$$

is close-to-convex in $|z| < 1$. When n is even (6) shows that $2\mu'_n(-1) = 2\sigma'_n(-r_n) = \phi_n(r_n)/(1+r_n)^3 = 0$ and hence the number r_n cannot be replaced by any larger one. The proof of Theorem 3 is therefore complete.

Theorem 4—The constant r_n, defined by Theorem 3, is such that

$$(2n^2+1)^{-\frac{1}{n}} \leqslant r_n \leqslant \left[\frac{n(n+1)}{2}\right]^{-\frac{1}{n-1}} \qquad \ldots \qquad \ldots \qquad (8)$$

and

$$r_n = 1 - \frac{\log(2n^2)}{n} + \frac{2\log(n)\log(2ne)}{n^2} + o\left(\frac{\log(n)}{n^2}\right) \qquad \ldots \qquad (9)$$

PROOF : We have

$$\phi_n\left((2n^2+1)^{-\frac{1}{n}}\right) = \frac{n\sqrt{n+1}}{2n^2+1}\left[\frac{n-1}{n+1} - \frac{1}{(2n^2+1)^{\frac{1}{n}}}\right]\left[3 + \frac{1}{(2n^2+1)^{\frac{1}{n}}}\right]$$

and hence $\phi_n((2n^2+1)^{-\frac{1}{n}}) \geqslant 0$ follows from the fact that

$$\left(\frac{n-1}{n+1}\right)^n \geqslant \frac{1}{2n^2+1},\ n \geqslant 2.$$

Thus we conclude that $r_n \geqslant (2n^2+1)^{-\frac{1}{n}}$. Further, it is well-known that if a polynomial $z + \sum_{k=2}^{n} a_k z^k$ is univalent in $|z| < 1$, then $|a_n| \leqslant 1/n$. But

$$\frac{1}{r_n}\sigma(r_n z) = \sum_{k=1}^{n} \frac{(k+1)}{2} r_n^{k-1} z^k$$

is univalent in $|z| < 1$ and hence $r_n \leqslant [n(n+1)/2]^{-\frac{1}{n}+1}$. This completes the proof of inequality (8).

In order to establish (9) we see that $\phi_n(r)=0$ yields

$$r_n = (2n^2)^{-\frac{1}{n}}\left[1 - \frac{(1-r_n)\ (2n+3)}{2n^2+4n+1}(1 + o(1)) + o(n^{-2})\right]. \quad .. \quad (10)$$

Since $0 < r_n < 1$, we have that

$$r_n = (2n^2)^{-\frac{1}{n}} + o(n^{-1}),$$

from which (9) follows.

Proceeding as in Theorem 3, we easily obtain the following;

Theorem 5—If $g(z) = z + a_2z^2 + \ldots \in K_{2\lambda}$, then each partial sum

$$s_n(z) = z + \sum_{d+2}^{n} a_k z^k \text{ of } g(z),$$

is close-to-convex/starlike/convex in the disc which is mapped onto a close-to-convex/star-like/convex domain, respectively, by the n-th partial sum, $z + \sum_{k=2}^{n} [\lambda + (1-\lambda)\,k]\, z^k$, of the function $g_1(z)$, defined by (5).

We remark that for $\lambda=1$, Theorem 5 yields the known results (Ruscheweyh 1972*a*, Ruscheweyh and Sheil-Small 1973).

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