ON THE EXPANSION OF *P*-FUNCTION

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In this paper an effort has been made to establish an expansion formula for the *P*-function in the form of its infinite series. An integral involving above function has also been evaluated.

The classical Laplace transform is given by

$$\phi(p) = p \int_0^\infty e^{-px} f(x)\, dx \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

and is represented by

$$\phi(p) = L\,[f(x)]. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Singh (1970) has introduced the *H*-transform as

$$\phi(p) = \lambda p \int_0^\infty \mathrm{H}^{m,n}_{r,t} \left[cpx \,\middle|\, \begin{matrix} \{(a_r, A_r)\} \\ \{(b_t, B_t)\} \end{matrix} \right] f(x)\, dx \qquad \ldots \qquad \ldots \qquad (3)$$

which is symbolically denoted by

$$\phi(p) = \mathrm{H}^{m,n}_{r,t} \left[f(x) : \lambda : \{(a_r, A_r)\} ; \{(b_t, B_t)\} \right]$$

where $\mathrm{H}^{m,n}_{r,t} \left[x \,\middle|\, - \right]$ is *H*-function defined by Fox (1961).

The aim of the present paper is to establish a relation between two functions which are transform pairs with respect to different kernels.

It is given in the form of the following theorem.

Theorem—If

$$\psi(p) = L\left[e^{-\eta x} x^{\mu-2}\, \mathrm{H}^{m_1,n_1}_{r_1,t_1} \left[ax \,\middle|\, \begin{matrix} \{(c_{r_1}, C_{r_1})\} \\ \{(d_{t_1}, D_{t_1})\} \end{matrix} \right] f(x) \right] \qquad \ldots \qquad (4)$$

$$f(p) = L\,[h(x)] \qquad (5)$$

and

$$p^{\rho}\, h(p) = \mathrm{H}^{m,\,n}_{r,\,t}\left[\, g(x) : \lambda\ \{(a_r\,, A_r)\}\ ;\ \{(b_t\,, B_t)\}\,\right] \qquad (6)$$

then

$$\psi(p) = \frac{\lambda p}{(p+\eta)^{l+\mu-2}} \int_0^\infty g(u)$$

$$\mathrm{P}\left[\begin{array}{l|l|l} \begin{bmatrix} 1,0 \\ 0,0 \end{bmatrix} & (l+\mu-2,1) & \dfrac{a}{p+\eta} \\ \begin{bmatrix} n_1, m_1 \\ r_1 - n_1, t_1 - m_1 \end{bmatrix} & \{(c_{r_1}\,, C_{r_1})\}\ ;\ \{(d_{r_1}\,, D_{r_1})\} & \\ \begin{bmatrix} m, n+1 \\ t-m,\ r-n \end{bmatrix} & \begin{array}{l}\{(1-b_t\,, B_t)\ ;\ \{(2-l,\,1)\}\,,\\ \{(1-a_r\,, A_r)\}\end{array} & \dfrac{1}{cu\,(p+\eta)} \end{array}\right] du \qquad (7)$$

provided that $\mathrm{Re}(p) > 0$, $\mathrm{Re}\,(\eta) > 0$, $a \geqslant 0$

$$\mathrm{Re}\left[\,2 - l + \min\left\{\frac{b_i}{B_j}\right\}\,\right] > 0,\ 1 \leqslant j \leqslant \mathrm{m}$$

$$\mathrm{Re}\left[\,l + \mu - 2 + \min\left(\frac{d_j}{D_j}\right) + \min\left(-l\,,\ \frac{1-a_j}{A_j}\right)\right] > 0$$

$$1 \leqslant j \leqslant m_1 \qquad 1 \leqslant j \leqslant n$$

$$\sum_{j=1}^{n} A_j - \sum_{j=n+1}^{r} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{t} B_j \equiv M^* > 0$$

$$\sum_{j=1}^{n_1} C_j - \sum_{j=n_1+1}^{r_1} C_j + \sum_{j=1}^{m_1} D_j - \sum_{j=m_1+1}^{t_1} D_j = N^* > 0$$

and

M^* and N^* have these values throughout this paper.

$$P\left[\begin{matrix}\begin{bmatrix}1,0\\0,0\end{bmatrix}\\ \begin{bmatrix}n_1, m_1\\ r_1-n_1\,,\ t_1-m_1\end{bmatrix}\\ \begin{bmatrix}m, n+1\\ t-m\,,\ r-n\end{bmatrix}\end{matrix}\ \middle|\ \begin{matrix}(l+\mu-2,1)\\ \{(c_{r_1}, C_{r_1})\}\ ;\ \{(d_{t_1}, D_{t_1})\}\\ \{(1-b_t, B_t)\}\ ;\ (2-l,1),\\ \{(1-a_r, A_r)\}\end{matrix}\ \middle|\ \begin{matrix}\dfrac{a}{p+\eta}\\ \\ \dfrac{1}{cu(p+\eta)}\end{matrix}\right]$$

$$=\frac{(p+\eta)^{\mu+l-2}}{p^{\mu+l-2}}\sum_{s=0}^{\infty}\frac{(-\eta)^s}{\lfloor s\, p}$$

$$P\left[\begin{matrix}\begin{bmatrix}1,0\\0,0\end{bmatrix}\\ \begin{bmatrix}n_1, m_1\\ r_1-n_1,\ t_1-m_1\end{bmatrix}\\ \begin{bmatrix}m, n+1\\ t-m,\ r-n\end{bmatrix}\end{matrix}\ \middle|\ \begin{matrix}(l+\mu+s-2,1)\\ \{(c_{r_1}, C_{r_1})\}\ ;\ \{(d_{t_1}, D_{t_1})\}\\ \{(1-b_t, B_t)\}\ ;\ (2-l,1),\\ (1-a_r, A_r)\end{matrix}\ \middle|\ \begin{matrix}\dfrac{a}{p}\\ \\ \dfrac{1}{cup}\end{matrix}\right] \quad .. \quad (8)$$

under the conditions of the theorem.

PROOF : From (5) and (6), we have

$$f(p)=p\int_0^{\infty} e^{-px}\left\{\lambda x^{1-l} \int H_{r,t}^{m,n}\left[cux\,\middle|\,\begin{matrix}\{(a_r, A_r)\}\\ \{(b_t, B_t)\}\end{matrix}\right] g(u)\,du\right\}dx \quad .. \quad (9)$$

Changing the order of integration which is permissible by De la Vallee Poussin's theorem, we get

$$f(p)=\lambda p^{l-1}\int_0^{\infty} H_{r+1,t}^{m,n+1}\left[\frac{cu}{p}\,\middle|\,\begin{matrix}(l-1,1),\{(a_r, A_r)\}\\ \{(b_t, B_t)\}\end{matrix}\right] g(u)\,du \quad .. \quad (10)$$

where we have used the known result due to Gupta and Jain (1968). Also

$$\psi(p)=p\int_0^{\infty} e^{-(p+\eta)x}\, H_{r_1,t_1}^{m_1,n_1}\left[ax\,\middle|\,\begin{matrix}\{(c_{r_1}, C_{r_1})\}\\ \{(d_{t_1}, D_{t_1})\}\end{matrix}\right] f(x)\,dx \quad .. \quad (11)$$

Substituting the value of $f(x)$ from (10) in (11), changing the order of integration and applying the known result due to Pathak (1970) in the evaluation of integral, we get

$$\psi(p) = \frac{\lambda p}{(p+\eta)^{l+\mu-2}} \int_0^\infty g(u)$$

$$P\left[\begin{matrix} \begin{bmatrix} 1, 0 \\ 0, 0 \end{bmatrix} (l+\mu-2, 1) \\ \begin{bmatrix} n_1, m_1 \\ r_1-n_1, \; t_1-m_1 \end{bmatrix} \{(c_{r_1}, C_{r_1})\} \; ; \; \{(d_{t_1}, D_{t_1})\} \\ \begin{bmatrix} m, n+1 \\ t-m, r-n \end{bmatrix} \{(1-b_t, B_t)\} \; ; \; (2-l, 1), \{(1-a_r, A_r)\} \end{matrix} \;\middle|\; \begin{matrix} \dfrac{a}{p+\eta} \\ \\ \dfrac{1}{cu\,(p+\eta)} \end{matrix}\right] du \tag{12}$$

If we write

$$\psi(p) = p \int_0^\infty e^{-(p+\eta)x} \, x^{\mu-2} \, f(x)\, dx$$

in the form

$$\psi(p) = p \int_0^\infty e^{-px} \sum_{s=0}^\infty \frac{(-\eta)^s}{\underline{|s}} \, x^{\mu+s-2} \, f(x)\, dx$$

we get

$$\psi(p) = \lambda p \int_0^\infty g(u)\, du \sum_{s=0}^\infty \frac{(-\eta)^s}{\underline{|s}} \int_0^\infty e^{-px} \times$$

$$\times \, x^{\mu+l+s-3} \, H^{m_1, n_1}_{r_1, t_1}\left[ax \;\middle|\; \begin{matrix} \{(a_{r_1}, A_{r_1})\} \\ \{(b_{t_1}, B_{t_1})\} \end{matrix} \right] \times$$

$$\times \, H^{n+1, m}_{t, r+1}\left[\frac{x}{cu} \;\middle|\; \begin{matrix} \{1-b_t, B_t)\} \\ (-l, 1), \{(1-a_r, A_r)\} \end{matrix} \right] dx \quad \ldots \quad \ldots \tag{13}$$

Evaluating the x integrals by a known result (Pathak 1970), we get

$$\psi(p) = \frac{\lambda p}{p^{\mu+l+s-2}} \int_0^\infty g(u) \sum_{s=0}^\infty \frac{(-\eta)^s}{\lfloor s} \times$$

$$\times P\left[\begin{array}{c|l|c} \begin{bmatrix} 1, 0 \\ 0, 0 \end{bmatrix} & (l+\mu+s-2, 1) & \\ \begin{bmatrix} n_1, & m_1 \\ r_1-n_1, & t_1-m_1 \end{bmatrix} & \{(c_{r_1}, C_{r_1})\} \; ; \; \{(d_{t_1}, D_{t_1})\} & \frac{a}{p+\eta} \\ \begin{bmatrix} m, & n+1 \\ t-m, & r-n \end{bmatrix} & \{(1-b_t, B_t)\}; (2-l, 1), \{(1-a_r, A_r)\} & \frac{1}{cu(p+\eta)} \end{array}\right] du \tag{14}$$

We have, from (2·5) and (2·3)

$$\lambda p \int_0^\infty g(u) \left\{ \frac{1}{(p+\eta)^{\mu+l-2}} \times \right.$$

$$\times P\left[\begin{array}{c|l|c} \begin{bmatrix} 1, 0 \\ 0, 0 \end{bmatrix} & (l+\mu-2, 1) & \\ \begin{bmatrix} n_1, & m_1 \\ r_1-n_1, & t_1-m_1 \end{bmatrix} & \{(c_{r_1}, C_{r_1})\} \; ; \; \{(d_{t_1}, D_{t_1})\} & \frac{a}{p+\eta} \\ \begin{bmatrix} m, & n+1 \\ t-m, & r-n \end{bmatrix} & \{(1-b_t, B_t)\} \; ; \; (2-l, 1,) \; \{(1-a_r, A_r)\} & \frac{1}{cu(p+\eta)} \end{array}\right] -$$

$$- \sum_{s=0}^\infty \frac{(-\eta)^s}{\lfloor s \; p^{\mu+l+s-2}}$$

$$\times P\left[\begin{array}{c|l|c} \begin{bmatrix} 1, 0 \\ 0, 0 \end{bmatrix} & (l+\mu+s-2, 1). & \\ \begin{bmatrix} n_1, & m_1 \\ r_1-n_1, & t_1-m_1 \end{bmatrix} & \{(c_{r_1}, C_{r_1},)\} : \{(d_{t_1}, D_{t_1})\} & \frac{a}{p} \\ \begin{bmatrix} m, & n+1 \\ t-m, & r-n \end{bmatrix} & \{(1-b_t, B_t)\} \; ; \; (2-l, 1), \; \{(1-a_r, A_r)\} & \frac{1}{cup} \end{array}\right] \times du = 0 \tag{15}$$

If we take $g(u) = e^{-\nu u} . u^{\alpha}$, Re $(\nu) > 0$, we see that the integral (15) is true for all values of η.

Since (15) holds for the general value of the parameter which occurs in the integrand only as the exponent of u with respect to which the integration is performed. It follows by the modification of the Lerch (1903) theorem that the integrand vanishes identically. Hence we get the formula (8).

Particular Case

Since the P-function is the very general function of two variables hence the expansion formulae for the most of special functions can be derived as its particular cases.

The following particular case due to Pande (1971) can be derived by taking $(C_{r_1}) = (D_{t_1}) = 1$, $(A_r) = (B_t) = 1$, $C = \lambda = 1$, $n_1 = r_1 = 0$, $m = t = 2$, $n = 0$, $r = 1$, $b_1 = 2m$, $b_2 = 0$, $a_1 = m - k + \frac{1}{2}$ and by replacing l by $2—l$, then by taking limit as $a \to 0$,

" If

$$\psi(p) = L\ [e^{-\eta x}\ x^{\mu-2}\ f(x)]$$

$$f(p) = L\ [h(x)]$$

$$p\ h(p) \underset{k,\ m}{\overset{2-l\ \ \frac{1}{2},\ m-\frac{1}{2}}{\rightleftarrows}} g(x)$$

then

$$\psi(p) = \frac{p}{(p+\eta)^{\mu}} \int_0^{\infty} u^{-l} \times$$

$$\times\ E\ [l, \mu, l + 2m,\ l + m - k + \tfrac{1}{2},\ (p+\eta)u]\ g(u)\ du \quad \ldots \quad (16)$$

and

$$E\ [l, l+2m,\ ;\ l+m-k+\tfrac{1}{2};\ (p+\eta)\ u]$$

$$= (p+\eta)^{\mu} \sum_{s=0}^{\infty} \frac{(-\eta)^s}{\lfloor s\ p^{\mu+s}} E\ [l, l + 2m,\ \mu + s;\ l + m - k + \tfrac{1}{2},\ pu] \quad \ldots \quad (17)$$

Example—If we take

$g(x) = x^b\ {}_2F_1\ (h_1,\ h_2;\ g;\ a\ x)$

then using the result due to Gupta and Mittal (1967) we get

$$h(p) = \frac{p^{1-l}\, a^{-b-1}}{\Gamma(h_1)\ \Gamma(h_2)}\ H^{m+2,\, n+1}_{r+2,\, t+2}\left[\frac{cp}{a}\ \middle|\ \times \right.$$

$$\left. \begin{matrix} (-b, 1), (g-b-1, 1), \{(a_r, A_r)\} \\ (h_1-b-1, 1), (h_2-b-1, 1), \{(b_t, B_t)\} \end{matrix} \right] \quad \ldots \quad \ldots \quad (18)$$

provided that $\mathrm{Re}\left[(b + 1 + \min\left(\frac{b_j}{B_j}\right)\right] > 0$

$$1 \leqslant j \leqslant m$$

$$\mathrm{Re}\left[b + 1 + \max\left(\frac{a_j - 1}{A_j}\right) - \min(h_i) \right] > 0$$

$$1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant 2$$

$M > 0,\ |\arg \alpha| < \pi,\ |\arg p| < \frac{1}{2}\pi M,$

therefore, with the help of the known result due to Gupta and Jain (1968), we have

$$f(p) = \frac{\lambda\, \Gamma(g)\ a^{-b-1}}{\Gamma(h_1)\ \Gamma(h_2)}\ H^{m+2,\, n+2}_{r+3,\, t+2}\left[\frac{c}{\alpha p}\ \middle|\ \times \right.$$

$$\left. \begin{matrix} (l-1, 1)\ (-b, 1), (g-b-1, 1), \{(a_r, A_r)\} \\ (h_1-b-1, 1), (h_2-b-1, 1), (b_t, B_t) \end{matrix} \right] \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

provided that $\mathrm{Re}\left[(2-l+\min\left\{ h_i-b-1, \left(\frac{b_j}{B_j}\right)\right\} \right] > 0;\ 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant m$

$|\arg \alpha| < \frac{1}{2}\pi M,\ M > 0.$

Hence

$$\psi(p) = \frac{\lambda\, \Gamma(g)\, p\, a^{-b-1}}{\Gamma(h_1)\ \Gamma(h_2)} \int_0^\infty e^{-(p+\eta)x}\, x^{l+\mu-3} \times$$

$$\times\ H^{m_1,\, n_1}_{r_1,\, t_1}\left[ax \ \middle|\ \begin{matrix} \{(c_{r_1}, C_{r_1})\} \\ \{(d_{t_1}, D_{t_1}\} \end{matrix} \right] \times$$

$$\times\ H^{n+2,\, m+2}_{t+2,\, r+3}\left[\frac{ax}{c}\ \middle|\ \begin{matrix} (2+b-h_1, 1), (2+2-h_2, 1), \{(1-b_t, B_t)\} \\ (2-l, 1), (1+b, 1), (2+b-g, 1), \{(1-a_r, A_r)\} \end{matrix} \right] dx \quad (20)$$

Now evaluating the above integral by the help of a known result due to Pathak (1970), and using the theorem we get the following integral

$$\int_0^\infty u^b\, {}_2F_1\,(h_1, h_2; g; \alpha\, u) \times$$

$$\times\, \mathrm{P}\left[\begin{matrix}\begin{bmatrix}1,0\\0,0\end{bmatrix}\\ \begin{bmatrix}n_1, & m_1\\ r_1-n_1, & t_1-m_1\end{bmatrix}\\ \begin{bmatrix}m, & n+1\\ t-m, & r-n\end{bmatrix}\end{matrix}\left|\begin{matrix}(l+\mu-2,1)\\ \{(c_{r_1}, C_{r_1})\};\{(d_{t_1}, D_{t_1})\}\\ \{(1-b_t, B_t)\};\,(2-l,1),\{(1-a_r, A_r)\}\end{matrix}\right|\begin{matrix}\dfrac{a}{p+\eta}\\ \dfrac{1}{cu(p+\eta)}\end{matrix}\right] du$$

$$= \frac{\Gamma\,(g)\;\alpha^{-b-1}}{\Gamma\,(h_1)\;\;\Gamma\,(h_2)}\;\mathrm{P}\left[\begin{matrix}\begin{bmatrix}1,0\\0,0\end{bmatrix}\\ \begin{bmatrix}n_1, & m_1\\ r_1-n_1, & t_1-m_1\end{bmatrix}\\ \begin{bmatrix}m+2, & n+2\\ t-m, & r-n+1\end{bmatrix}\end{matrix}\right.$$

$$\left.\left|\begin{matrix}(l+\mu-2,1)\\ \{(c_{r_1}, C_{r_1})\};\ \{(d_{t_1}, D_{t_1})\}\\ (2+b-h_1\ 1), (2+b-h_2, 1)\ \{(1-b_t, B_t)\};\\ (2-l,1)\ (l+b,\ 1), (2+b-g,\ 1),\ \{(l-a_r,\ A_r)\}\end{matrix}\right|\begin{matrix}\dfrac{a}{p+\eta}\\ \dfrac{a}{c(p+\eta)}\end{matrix}\right] \tag{21}$$

provided that $\mathrm{Re}\,(p) > 0$, $\mathrm{Re}\,(\eta) > 0$, $\mathrm{Re}\left[\,b+1+\min\left(\dfrac{b_j}{B_j}\right)\right] > 0,$
$1 \leqslant j \leqslant m$

$\mathrm{Re}\left[\,(b+1+\max\left(\dfrac{a_j-1}{A_j}\right)\right] - \min\,(h_r\,)\;0,\ 1 \leqslant j \leqslant m,\ 1 \leqslant r \leqslant 2$

$\mathrm{Re}\left[\,2-l+\min\left\{h_i-b-1, \left(\dfrac{b_j}{B_j}\right)\right\}\right] > 0,\ 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant m$

$\mathrm{Re}\left[\,l+\mu-2+\min\left(\dfrac{d_i}{B_i}\right)+\min\,\{2-l,\ 1+b,\ 2+b-g,\right.$
$\left.\left(\dfrac{1-a_j}{\ }\right)\right\}\Big] > 0,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n$

and $|\arg \alpha| < \min(\pi, \frac{1}{2}\pi M)$. And from (21) and (15) we have

$$
P\left[\begin{array}{c}\left[\begin{array}{l}1\,,\,0\\0\,,\,0\end{array}\right]\\\left[\begin{array}{l}n_1\,,\,m_1\\r_1-n_1\,,\,t_1-m_1\end{array}\right]\\\left[\begin{array}{l}m+2\,,\,n+2\\ \\t-m,\;r-n+2\end{array}\right]\end{array}\left|\begin{array}{l}(l+\mu-2,1)\\\{(c_{r_1},C_{r_1})\}\;;\;\{(d_{t_1},D_{t_1})\}\\(2+b-h_1,1,)\;(2+b-h_2,1),\\\qquad\{(1-b_t,B_t)\}\;;\\(2-l,1),\;(1+b,1),\;(2+b-g,1),\\\qquad\{(1-a_r,A_r)\}\end{array}\right|\begin{array}{c}\dfrac{a}{p+\eta}\\ \\ \dfrac{a}{c(p+\eta)}\end{array}\right]
$$

$$
=\frac{(p+\eta)^{\mu+l-2}}{p^{\mu+l-2}}\sum_{S=0}^{\infty}\frac{(-\eta)^s}{\lfloor S\;p^s}\times
$$

$$
P\left[\begin{array}{c}\left[\begin{array}{ll}1, & 0\\0, & 0\end{array}\right]\\\left[\begin{array}{l}n_1,\;m_1\\r_1-n_1,\;t_1-m_1\end{array}\right]\\\left[\begin{array}{l}m+2,\;n+2\\t-m,\;r-n+2\end{array}\right]\end{array}\left|\begin{array}{l}(l+\mu+s-2,1)\\ \\\{(c_{r_1},C_{r_1})\}\;;\;\{(d_{t_1},D_{t_1})\}\\(2+b-h_1,1),\;(2+b-h_2,1),\;\{(1-b_t,B_t)\}\;;\\(2-l,1),\;(1+b,1),\;(2+b-g,1),\;\{(1-a_r,A\;)\end{array}\right|\begin{array}{c}\dfrac{a}{p}\\ \\ \dfrac{a}{Cp}\end{array}\right]
$$

provided the conditions of (21) are satisfied.

Giving particular values to the parameter in (19) and (21) one can get a number of results as its particular cases.

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