ON CONHARMONIC CURVATURE TENSOR

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Adati and Maiyazawa (1967) have studied conformally symmetric and recurrent spaces. In the present paper we have studied conharmonically symmetric and recurrent spaces in Kähler space.

1. INTRODUCTION

Let us consider a $2n$-dimensional space M_{2n} of differentiability C^∞. Let there be defined in M_{2n} a vector valued linear function F such that

$$\bar{\bar{X}} + X = 0 \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}1)$$

for arbitrary vector field X, where

$$\bar{X} = F(X). \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}2)$$

Then F is said to give an almost complex structure on M_{2n} and M_{2n} is said to be an almost complex space.

Agreement (1·1)—In what follows in this paper the equations containing X, Y, Z, U, V hold for arbitrary vector fields X, Y, Z, U, V.

Let the almost complex space M_{2n} be also endowed with the Hermitian metric g:

$$g(\bar{X}, \bar{Y}) = g(X, Y). \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}3)$$

Then M_{2n} is called an almost Hermite space.

Suppose D is the Riemannian connexion in M_{2n}. Then, if $[,]$ denotes the Lie bracket, we have

$$D_X Y - D_Y X = [X, Y] \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}4)$$

$$X\{g(Y, Z)\} = g(D_X Y, Z) + g(Y, D_X Z). \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}5)$$

If in addition to (1·2), (1·3), (1·4) and (1·5), we also have

$$(D_X F)(Y) = 0, \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}6)$$

then M_{2n} is called a Kähler space.

Let K be the curvature tensor of M_{2n}. Then

$$K(X, Y, Z) \overset{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z. \qquad .. \qquad .. \qquad .. \qquad (1{\cdot}7)$$

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It is well known (Adati and Miyazawa 1967) that M_{2n} is a recurrent space if

$$(D_V K)(X, Y, Z) = B(V) K(X, Y, Z) \quad .. \quad .. \quad (1{\cdot}8)$$

where B is a 1—form

Let Ric be the Ricci tensor. The space M_{2n} is called Ricci recurrent if

$$(D_V \text{ Ric})(X, Y) = B(V) \text{ Ric } (X, Y). \quad .. \quad .. \quad (1{\cdot}9)$$

From (1·9), we have

$$(D_V r)(X) = B(V) r(X) \quad .. \quad .. \quad .. \quad (1{\cdot}10)$$

and

$$D_V K' = B(V) K' \quad .. \quad .. \quad .. \quad (1{\cdot}11)$$

where

$$g(r(X), Y) = \text{Ric } (X, Y)$$

and K' is the scalar curvature.

The conharmonic curvature tensor L, Weyl's projective curvature tensor W and H-projective curvature tensor P are given by

$$L(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(n-1)} [g(Y, Z) r(X) - g(X, Z) r(Y) + \text{Ric } (Y, Z) X - \text{Ric } (X, Z) Y] \quad .. \quad .. \quad .. \quad (1{\cdot}13)$$

$$W(X, Y, Z) = K(X, Y, Z) + \frac{1}{2n-1} [\text{Ric } (X, Z) Y - \text{Ric } (Y, Z) X] \quad (1{\cdot}14)$$

$$P(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(n-1)} [\text{Ric } (Y, Z) X - \text{Ric } (X, Z) Y + \text{Ric } (X, \bar{Z}) \bar{Y} - \text{Ric } (Y, \bar{Z}) \bar{X} + 2 \text{ Ric } (X, \bar{Y}) Z]. \quad .. \quad (1{\cdot}15)$$

The space M_{2n} is called 'conharmonically recurrent' if

$$(D_V L)(X, Y, Z) = A(V) L(X, Y, Z) \quad .. \quad .. \quad .. \quad (1{\cdot}16)$$

where A is a 1-form.

For simplicity we call such a space an LK_n — space.

The space will be called 'conharmonically symmetric' if

$$(D_V L)(X, Y, Z) = 0. \quad .. \quad .. \quad .. \quad (1{\cdot}17)$$

For simplicity the space will be called an LS_n space. The space M_{2n} is called projectively recurrent if

$$(D_U W)(X, Y, Z) = A(U) W(X, Y, Z). \quad .. \quad .. \quad .. \quad (1{\cdot}18)$$

The space M_{2n} is called H-projectively recurrent if

$$(D_V P)(X, Y, Z) = A(V) P(X, Y, Z). \quad .. \quad .. \quad .. \quad (1\cdot19)$$

The space M_{2n} is called H-projectively symmetric if

$$(D_V P)(X, Y, Z) = 0. \quad .. \quad .. \quad .. \quad (1\cdot20)$$

The space is called an Einstein space if

$$\text{Ric}(X, Y) = a\, g(X, Y) \quad .. \quad .. \quad .. \quad (1\cdot21)$$

where

$$a = \frac{K'}{2n}.$$

It is well known (Adati and Miyazawa 1967) that in an Einstein space

$$D_W K' = 0 \quad .. \quad .. \quad .. \quad (1\cdot22)$$

and

$$(D_W \text{Ric})(X, Y) = 0. \quad .. \quad .. \quad .. \quad (1\cdot23)$$

2. LK_n — Space

The recurrence parameters are taken to be the same for each proof of this section.

Theorem 2·1—An Einstein M_{2n} is recurrent if and only if it is an LK_n; further, it is symmetric if and only if it is an LS_n. In particular if Ric = 0 in an M_{2n}, then it is recurrent (symmetric) if and only if it is LK_n (LS_n).

Proof : In an Einstein space M_{2n}, the conharmonic curvature tensor L is given by

$$L(X, Y, Z) = K(X, Y, Z) - \frac{a}{n-1}\,[g(Y, Z)\,X - g(X, Z)\,Y]. \quad .. \quad (2\cdot1)$$

Differentiating (2·1) covariantly, we get

$$(D_W L)(X, Y, Z) = (D_W K)(X, Y, Z). \quad .. \quad .. \quad .. \quad (2\cdot2)$$

From (2·2) the first part of the theorem follows. In particular if Ric = 0 then from (1·13) we have

$$L(X, Y, Z) = K(X, Y, Z)$$

which yields the second part of the theorem.

Theorem 2·2—A projectively recurrent space is LK_n.

$$(D_U W)(X, Y, Z) = (D_U K)(X, Y, Z) + \frac{1}{2n-1}[D_U \text{Ric})(X, Z) Y - (D_U \text{Ric})(Y, Z) X] \quad .. \quad .. \quad .. \quad (2\cdot3)$$

Using (1·14) and (1·18), we get

$$(D_V K)(X, Y, Z) = B(V) K(X, Y, Z) \quad .. \quad .. \quad (2\cdot4)$$

$$(D_V \text{Ric})(X, Y) = B(V) \text{Ric}(X, Y). \quad .. \quad .. \quad (2\cdot5)$$

Thus a projectively recurrent space is recurrent and Ricci-recurrent. Now differentiating (1·13) and taking account of (2·10) and (2·11), we get

$$(D_V L)(X, Y, Z) = B(V) L(X, Y, Z)$$

hence the space is LK_n.

Theorem 2.3—For a Kähler space under the same recurrence parameters if any two of the following properties hold, third also holds good

(a) the space is LK_n

(b) the space is H-projectively recurrent

(c) the space is Ricci—recurrent.

PROOF : From (1·13) and (1·15) we have

$$L(X, Y, Z) = P(X, Y, Z) + \frac{1}{2(n+1)}\left[\text{Ric}(Y, Z) X - \text{Ric}(X, Z) Y + \text{Ric}(X, \bar{Z}) \bar{Y} - \text{Ric}(Y, \bar{X}) \bar{X} + 2 \text{Ric}(X, \bar{Y}) \bar{Z}\right] - \frac{1}{2(n-1)}\left[g(Y, Z) r(X) - g(X, Z) r(Y) + \text{Ric}(Y, Z) X - \text{Ric}(X, Z) Y\right]. \quad .. \quad .. \quad .. \quad (2\cdot6)$$

Differentiating (2·6) covariantly, we have

$$(D_V^{\wedge} L)(X, Y, Z) = (D_V^{\wedge} P)(X, Y, Z) + \frac{1}{2(n+1)}\left[(D_V^{\wedge} \text{Ric})(Y, Z) X - (D_V^{\wedge} \text{Ric})(X, Z) Y + (D_V^{\wedge} \text{Ric})(X, \bar{Z}) \bar{Y} - (D_V^{\wedge} \text{Ric})(Y, \bar{Z}) \bar{X} + 2 (D_V^{\wedge} \text{Ric})(X, \bar{Y}) \bar{Z}\right] - \frac{1}{2(n-1)}\left[g(Y, Z)(D_V^{\wedge} r)(X)\right.$$

$$-g(X,Z)(D_V^{\wedge} r)(Y) + (D_V^{\wedge} \text{Ric})(Y,Z)X$$

$$-(D_V^{\wedge} \text{Ric})(X,Z)Y \Big]. \qquad .. \qquad .. \qquad .. \qquad (2\cdot7)$$

In light of (2·6) and (2·7) the result follows.

3. LS_n — Space

For this section we have taken the non null and same recurrence parameters for each proof.

Theorem 3·1—Every recurrent LS_n space is conharmonically flat and a LS_n – space is Ricci – recurrent if and only if

$$(D_V K)(X,Y,Z) + B(V)(L(X,Y,Z) - K(X,Y,Z)) = 0.$$

Proof : Differentiating (1·13) covariantly, we have

$$(D_V L)(X,Y,Z) = (D_V K)(X,Y,Z) - \frac{1}{2(n-1)}\Big[g(Y,Z)(D_V r)(X)$$

$$-g(X,Z)(D_V r)(X) + (D_V \text{Ric})(Y,Z)X - (D_V \text{Ric}(X,Z)Y \Big]. \qquad (3\cdot1)$$

Since the space is LS_n and recurrent, from (1·13) we get

$$B(V)\ L(X,Y,Z) = 0. \qquad .. \qquad .. \qquad .. \qquad (3\cdot2)$$

From this we have

$$L(X,Y,Z) = 0, \qquad .. \qquad .. \qquad .. \qquad 3\cdot3)$$

which proves the result.

In order to prove the latter result let the LS_n – space be Ricci – recurrent. Then

$$(D_V K)(X,Y,Z) - \frac{1}{2(n-1)} B(V)\Big[g(Y,Z) r(X)$$

$$-g(X,Z) r(Y) + \text{Ric}(Y,Z)X - (\text{Ric})(X,Z)Y\Big] = 0. \qquad (3\cdot4)$$

Using (1·13) in (3·4) we get

$$0 = (D_V\ K)\ (X, Y, Z) + B(V)\ [L\ (X, Y, Z) - K\ (X,\ Y, Z)]. \quad .. \quad (3\cdot5)$$

Conversely, let (3·5) be true. Then using this in

$$(D_V\ K)\ (X, Y, Z) - \frac{1}{2(n-1)} \Big[g\ (Y, Z)\ (D_V\ r)\ (X) - g\ (X, Z)\ (D_V\ r)\ (Y)$$

$$+ (D_V\ \mathrm{Ric})\ (Y, Z)\ X - (D_V\ \mathrm{Ric})\ (X, Z)\ Y \Big] = 0 \quad (3\cdot6)$$

we get

$$(D_V\ \mathrm{Ric})\ (X, Y) = B(V)\ \mathrm{Ric}\ (X, Y) \quad .. \quad .. \quad .. \quad (3\cdot7)$$

hence the result follows.

Theorem 3·2—The recurrent Einstein LS_n space is flat.

PROOF : In Einstein space, we have

$$(D_V^{\wedge}\ L)\ (X, Y, Z) = (D_V^{\wedge}\ K)\ (X, Y, Z).$$

Since the LS_n –space is recurrent, we have

$$B(V)\ K(X, Y, Z) = 0,$$

which implies that the space is flat.

Theorem 3·3—If the LS_n - space is H-projectively recurrent and Ricci-recurrent under the same recurrence parameter, then the space is conharmonically flat.

PROOF : Putting zero equal to the left hand side of (2·7) and using the fact that the space is H-projectively recurrent and Ricci-recurrent under the same recurrence parameter, we have

$$P\ (X, Y, Z) + \frac{1}{2(n+1)} \Big[\mathrm{Ric}\ (Y, Z)\ X - \mathrm{Ric}\ (X, Z)\ Y + \mathrm{Ric}\ (X, \bar{Z})\ \bar{Y}$$

$$- \mathrm{Ric}\ (Y, \bar{Z})\bar{X} + 2\ \mathrm{Ric}\ (X, \bar{Y})\bar{Z} \Big] - \frac{1}{2(n-1)} \Big[g\ (Y, Z)\ r\ (X) - g(X, Z)\ r\ (Y)$$

$$+ \mathrm{Ric}\ (Y, Z)\ X - \mathrm{Ric}\ (X, Z)Y \Big] = 0. \quad .. \quad .. \quad (3\cdot8)$$

In the light of (3·8) and (2·6), the result follows.

Theorem 3·4—If the space is H-projectively symmetric, LK_n and Ricci-recurrent under the same recurrence parameter then the space is H-projectively flat.

Proof follows the pattern of that of Theorem (3·3).

Theorem 3·5—If the LS_n —space is H-projectively symmetric and is Ricci-recurrent under the same recurrence parameters then the conharmonic curvature tensor and H-projective tensor coincide.

PROOF : Using the facts given in the theorem, we have in the light of (2·17)

$$\frac{1}{2(n+1)}\Big[\text{Ric}(Y,Z)X - \text{Ric}(X,Z)Y + \text{Ric}(X,\bar{Z})\bar{Y} - \text{Ric}(Y,\bar{Z})\bar{X} + 2\text{Ric}(X,\bar{Y})\bar{Z}\Big] - \frac{1}{2(n-1)}\Big[g(Y,Z)r(X) - g(X,Z)r(Y) + \text{Ric}(Y,Z)X - \text{Ric}(X,Z)Y\Big] = 0. \quad \ldots \quad (3\cdot9)$$

Putting (3·9) in (2·6) we obtain the required result.

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REFERENCE

Adati, A. and Miyazawa, T. (1967). On a Riemannian space with recurrent conformal curvature. *Tensor N. S.*, **18,** 355-42.