SEQUENCE OF MAPPINGS AND FIXED POINTS

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(Communicated by F. C. Auluck, F.N.A.)

(Received 7 May 1974)

Let X denote a metric space, ρ the metric on X and let H_1, be the family of functions $\alpha : (0, \infty) \to [0, 1)$ such that α is monotonically decreasing.

Theorem 1—For $n=0. 1, 2, \ldots$, let T_n be a sequence of mappings of a complete metric space$X$ such that for each $n = 0, 1, 2, \ldots, T_n$ satisfy $\rho(T_n x, T_n y)$
$$\leqslant \alpha (\rho (x,y)) \rho (x, T_n x) + \beta (\rho (x,y)) \rho (y, T_n y) \quad \ldots \quad (1)$$
for all x and y in X with $x \neq y$, $\alpha, \beta \in H_1$ such that $\alpha (t) + \beta (t) < 1, t \in (0, \infty)$. If a_n denotes the fixed point of $T_n , \ldots \ldots,$ and if $T_n \to T_0$ pointwise on X, then $a_n \to \alpha_n$.

Theorem 2—Let X be a complete metric space and for $n=1, 2, 3, \ldots$, let $T_n : X \to X$ be a sequence of functions with at least one fixed point a_n and let $T_0 : X \to X$ satisfy condition of Theorem 1 with same functions α and β.

If $T_n \to T_0$ uniformly on X then $a_n \to a_0$, the unique fixed point of T_0 .

Theorem 3—Let X be a locally compact metric space and let $T_i : X \to X$ be a contractive mapping with fixed point a_i for each $i = 1, 2, \ldots,$ and let $T_0 : X \to X$ satisfy

$\rho (T_0 x, T_0 y) \leqslant \alpha (\rho (x,y)) \rho (x,y)$ for all x and y in X with $x \neq y$, $\alpha \in H_1$, with fixed point a_0 . If $T_i \to T_0$ pointwise then $a_i \to a_0$.

INTRODUCTION

Let (X, ρ) be a metric space.

Definition 1—A mapping $T: X \to X$ is said to be contractive if for all x and y in X with $x \neq y$, $\rho (Tx, Ty) < \rho (x,y)$.

Definition 2— Let (X, ρ) be a metric space. A mapping $T: X \to X$ is said to be a contraction mapping if there exists a constant k, $0 \leqslant k < 1$, such that for every x and y in X,

$$\rho(Tx, Ty) \leqslant k \rho (x,y).$$

Definition 3—H_1 is defined to be the family of all functions $\alpha: R^+ \to [0, 1)$ such that α is monotonically decreasing. By the Banach contraction Theorem, it is known that a contraction mapping of complete metric space into itself has a unique fixed point. Rakotch (1962) proved that theorem of Banach remains true if the constant k is replaced by a function α in H_1. A contractive mapping of a complete metric space into itself need not have a fixed point. For example, if $X = \{x \mid x \geqslant 1\}$ and $T: X \to X$ be defined by $Tx = x + \frac{1}{x}$ then T has no fixed point. However, if T has a fixed point it will be always unique. In fact Edelstein (1962) proved the following theorem : If X is a compact metric space and $T : X \to X$ is a contractive mapping, then T has a unique fixed point. Bonsall (1962) asked if in a complete metric space, does the convergence of a sequence of contraction maps to a contraction map imply the convergence of their fixed points.

A partial answer is:

Theorem (Bonsall 1962) — Let X be a complete metric space and for $n = 0, 1, 2, \ldots$, let $T_n : X \to X$ be a sequence of contractions, all with the same constant $k < 1$. If a_n denotes the unique fixed point of T_n and $T_n \to T_0$ pointwise on X then $a_n \to a_0$. Nadler (1968) became interested in the same problem and asked whether the constant $k < 1$ was absolutely necessary for all the maps T_n. He found conditions where T_0 need be a contraction.

Theorem (Nadler 1968)—For $n = 1, 2, 3, \ldots$, let $T_n : X \to X$ (X is complete) be a sequence of functions such that each T_n has at least one fixed point a_n. If $T_0: X \to X$ is a contraction with fixed point a_0 and if $T_n \to T_0$ uniformly on X then $a_n \to a_0$.

In this paper we prove some theorems on sequence of contraction mappings which generalize the theorems of Bonsall (1962) and Nadler (1968). The following theorem has been proved by the author (Ray 1974).

Theorem 1—Let X be a complete metric space and let $T: X \to X$ satisfy

$$\rho(Tx, Ty) \leqslant \alpha(\rho(x,y))\, \rho(x, Tx) + \beta(\rho(x,y))\, \rho(y, Ty) \tag{1}$$

for all x and y in X with $x \neq y$

where $\alpha, \beta \in H_1$ such that $\alpha(t) + \beta(t) < 1$, $t \in (0, \infty)$. Then T has a unique fixed point.

We apply this result to the following propositions which are generalizations of the results of (Bonsall 1962 p. 6) and Nadler (1968).

Theorem 2— Let (X, ρ) be a complete metric space and for $n=0, 1, 2, \ldots$ let $T_n : X \to X$ satisfy the condition (1) with the same functions α and β. If a_n denotes the unique fixed point of T_n and $T_n \to T_0$ pointwise on X then $a_n \to a_0$.

Proof : By Theorem 1, each T_n has a unique fixed point a_n. Now

$$\begin{aligned} \rho(a_n, a_0) &= \rho(T_n a_n, T_0 a_0) \\ &\leqslant \rho(T_n a_n, T_n a_0) + \rho(T_n a_0, T_0 a_0) \\ &\leqslant \alpha(\rho(a_n, a_0))\, \rho(a_n, T_n a_n) + \beta(\rho(a_n, a_0))\, \rho(a_0, T_n a_0) \end{aligned}$$

$$+ \rho (T_n a_0, T_0 a_0),$$
$$= [1 + \beta(\rho (a_n , a_0))] \rho(T_n a_0, T_0 a_0)$$
$$\leqslant 2 \rho (T_n a_0, T_0 a_0).$$

Let $\epsilon > 0$ be given. There is an N such that for $n \geqslant N$

$$\rho(T_n a_0, T_0 a_0) < \frac{\epsilon}{2}.$$

Hence $\rho (a_n , a_0) < \epsilon$ for $n \geqslant N$.

Thus $a_n \to a_0$.

Theorem 3—Let X be a complete metric space and for $n=1, 2, 3, \ldots,$ let $T_n: X \to X$ be a sequence of maps with at least one fixed point a_n and let $T_0: X \to X$ satisfy the condition (1) of Theorem 1 with the same functions α and β. If $T_n \to T_0$ uniformly and a_0 be the unique fixed point of T_0, then $a_n \to a_0$.

Proof : Let $\epsilon > 0$ be arbitrary and choose $\epsilon_1 > 0$ such that

$$\epsilon_1 < \frac{\epsilon_0}{1+\alpha(\epsilon_0)}. \qquad \ldots \quad \ldots \quad \ldots \qquad (2)$$

By the uniform convergence of the T_n there is a positive integer N such that for all $k \geqslant N$ and for all $x \in X$, $\rho(T_k x, T_0 x) < \epsilon_1$.

Claim : For all $i \geqslant N$, $\rho(a_i, a_0) < \epsilon_0$. Suppose not. Then there exists a $j \geqslant N$ such that

$$\rho (a_j, a_0) \geqslant \epsilon_0. \qquad \ldots \quad \ldots \quad \ldots \qquad (3)$$

But since $T_n \to T_0$ uniformly and $a_0 = T_0 a_0$, $a_j = T_j a_j$, $\rho (a_j , a_0)$

$$\leqslant \rho (T_j a_j , T_0 a_j) + \rho (T_0 a_j , T_0 a_0)$$
$$\leqslant \rho (T_j a_j , T_0 a_j) + \alpha (\rho (a_j, a_0)) \rho (a_j, T_0 a_j) + \beta (\rho (a_j, a_0)) \rho (a_0, T_0 a_0)$$
$$= [1+ \alpha (\rho (a_j, a_0))] \rho (T_j a_j, T_0 a_j)$$
$$< [1 + \alpha (\rho (a_j, a_0))] \epsilon_1 . \qquad \ldots \quad \ldots \quad \ldots \qquad (4)$$

However α is a monotonically decreasing function, which by (3) implies

$$\alpha (\rho (a_j, a_0)) \leqslant \alpha (\epsilon_0)$$

So $1 + (\alpha \rho (a_j, a_0)) \leqslant 1 + \alpha (\epsilon_0)$.

This, coupled with the choice of ϵ_1 and (4) implies $\rho (a_j, a_0) < \epsilon_0$ which is a contradiction of (3). Therefore $a_n \to a_0$.

Theorem 4. Let (X, ρ) be a locally compact metric space and let T_i: $X \to X$ be a contractive mapping with fixed point a_i
for each $i=1, 2, 3, ..,$ and let T_0: $X \to X$ satisfy $\rho(T_0x, T_0y) \leqslant \alpha(\rho(x,y))\ \rho(x,y)$ for all distinct x, y in X where $\alpha \in H_1$.

It a_0 is the unique fixed point of T_0 and if the sequence $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_0 then $a_i \to a_0$.

PROOF : If follows from a theorem of Rakotch (1962) that a_0 is a unique fixed point of T_0. Lef $\epsilon > 0$ and assume ϵ is sufficiently small so that

$$S(a_0, \epsilon) = \{x \in X \mid \rho(a_0, x) \leqslant \epsilon\}$$

is a compact subset of X. Then, since $\{T_i\}_{i=1}^{\infty}$ is an equicontinuous sequence of functions converging pointwise to T_0 and since $S(a_0, \epsilon)$ is compact, the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly on $S(a_0, \epsilon)$ to T_0.

Now $\rho(T_i x, a_0)$

$$\leqslant \rho(T_i x, T_0 x) + \rho(T_0 x, T_0 a_0)$$
$$\leqslant \rho(T_i x, T_0 x) + \alpha(\rho(x, a_0))\ \rho(x, a_0)$$
$$\leqslant \rho(T_i x, T_0 x) + \alpha\ \rho(x, a_0)$$

we choose $N>0$ such that for $i \geqslant N$ and for all $x \in S(a_0, \epsilon)$,

$\rho(T_i x, T_0 x) < (1-\alpha)\ \epsilon$.

So if $i \geqslant N$ and $x \in S(a_0, \epsilon)$, then

$\rho(T_i x, a_0) < (1-\alpha)\ \epsilon + \alpha\ \rho(x, a_0) \leqslant (1-\alpha)\ \epsilon + \alpha\ \epsilon = \epsilon.$

This proves that if $i \geqslant N$ then $\{T_i\}_{i=1}^{\infty}$ maps $S(a_0, \epsilon)$ into itself. Let T_i' be the restriction of T_i to $S(a_0, \epsilon)$. Then we observe that each T_i' is a contractive mapping of $S(a_0, \epsilon)$ into itself.

So, since $S(a_0, \epsilon)$ is compact, by the theorem of Edelstein (1962) each T_i' has a unique fixed point for each $i \geqslant N$ and from the definition of T_i' and the fact that each T_i has a unique fixed point, it follows that the unique fixed point of T_i' is a_i. So for each $i \geqslant N$, $a_i \in S(a_0, \epsilon)$.

Therefore the sequence, $\{a_i\}$ of fixed points converges to a_0.

A careful examination of the proof of Theorem 4 indicates that it was not necessary to assume $\rho(T_0x, T_0y) \leqslant \alpha(\rho(x, y))\ \rho(x, y)$ for all x and y in X but only

$$\rho(T_0x, a_0) \leqslant \alpha(\rho(x, a_0))\ \rho(x, a_0) \qquad \ldots \qquad \ldots \qquad (5)$$

for all x in X where a_0 is the unique fixed point of T_0.

Thus the following theorem is proved.

Theorem 5—Let X be a locally compact metric space and T_i: $X \to X$ be a contractive mapping with fixed point a_i for each $i=1, 2, 3, \ldots$. Let T_0: $X \to X$ be a function with a unique fixed a_0 such that for all x in X, (5) holds where α: $R^+ \to [0, 1)$ is a mono tonically decreasing function. Then if $\{T_i\}_{i=1}^{\infty}$ converges pointwise to T_0, then $a_i \to a_0$.

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