NEW CRITERIA FOR THE ABSOLUTE RIESZ SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES

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The author has proved the following :

Theorem 1—Let positive numbers $a,b,c,$ be such that

$$0 < b \leqslant 1, b = a\,(1+b) \text{ and } c(1+b) > 1. \qquad (\alpha)$$

Then

$$\phi\,(t) \log \frac{1}{t} \in BV\,(0, p), \text{ where } 0 < p < e^{-2}, \qquad (\beta)$$

implies that $\sum_{n=1}^{\infty} A_n\,(x) \in |\,R, \exp\{w^a / (\log w)^c\}, 1+b\,|$.

Theorem 2—Let (α) and (β) hold. Then, in order that $\sum_{n=}^{\infty} A_n\,(x) \in |\,C, 0\,|$, it is necessary and sufficient that $\{n^{1-a} A^n\,(x)\;(\log n)^c\} \in |\,R, \exp.\{w^a / (\log w)^c\}, 1\,|$.

The analogue of Theorems 1 and 2 for the conjugate series have also been obtained on replaeing (β) by the conditions

$$\psi\,(+0) = 0 \text{ and } \int_0^p \log \frac{1}{t}\,|\,d\,\psi\,(t)\,| < \infty .$$

1. DEFINITIONS AND NOTATIONS

Let f be $2\,\pi$ — periodic function and that L-integrable over $(-\pi, \pi)$. Let, without loss of generality, its Fourier series, at a point $t = x$, be given by

$$\sum (a_n \cos nx + b_n \sin nx) = \sum A_n\,(x).\,* \qquad (1{\cdot}1)$$

* Summations, with respect to n, are over 1, 2, 3,....∞, when there is no indication to the contrary.

Then the conjugate series of (1·1) will be given by

$$\sum (b_n \cos nx - a_n \sin nx) = \sum B_n (x) .$$

Throughout the paper a,b and c are positive numbers. Also, for the convenience, we use the following notations :

$$\phi (t) = \tfrac{1}{2} \{f (x + t) + f (x - t)\} \qquad (1\cdot2)$$

$$\psi (t) = \tfrac{1}{2} \{f (x + t) - f (x - t)\} \qquad (1\cdot3)$$

$$e (w) = \text{exn} \{w^a/(\log w)^c\} \text{ for } w > 1 \text{ and } e (1) = 1 \qquad (1\cdot4)$$

$$e^q (w) = (e (w))^q , \text{ for finite } q \qquad (1\cdot5)$$

$$F^{(1)} (w) = \frac{d}{dw}(F (w)) \qquad (1\cdot6)$$

$$K (w, t) = \sum_{n\leqslant w} (e (w) - e (n))^b e (n) \cos nt \qquad (1\cdot7)$$

$$\tilde{K}(w, t) = \sum_{n\leqslant w} (e (w) - e (n))^b e(n) \sin nt \qquad (1\cdot8)$$

$$E (w, t) = \sum_{n\leqslant w} (e (w) - e (n))^b e(n) \frac{\sin nt}{n} \qquad (1\cdot9)$$

$$\bar{E} (w, t) = \sum_{n\leqslant w} (e (w) - e (n))^b e (n) \frac{\cos nt}{n} \qquad (1\cdot10)$$

2. Introduction

Concerning the absolute Riesz summability of a Fourier series and its conjugate series, at a point $t = x$, of the type $\exp(w^a)$ $(0 < a < 1)$, the following theorems are known :

Theorem A (Mohanty 1950, *Theorem B)* — Let $k > \pi e$. Then

$$\phi (t) \log \frac{k}{t} \in BV (0, \pi)^{+} \qquad (2\cdot1)$$

implies that $\sum A_n (x) \in | R, \exp (w^a), | \ (0 < a < 1)$.

+ $f (t) \in B V (a, b)$ means that $f (t)$ belong to the class cf functions of bounded variation over (a, b).

‡ $\Sigma\ a_n \in | A |$ means that $\Sigma\ a_n$ belongs to the class of series summable by the method $| A |$.

Theorem B (Mohanty 1951, *Theorem 5A)*—Let $k > \pi e$. Then $\psi(+0) = 0$ and

$$\int_0^{\pi} \log \frac{k}{t} \mid d\psi(t) \mid < \infty \text{ imply that } \sum B_n(x) \in \mid R, \exp(w^a), \mid \; (0 < a < 1).$$

Replacing the non-local conditions, imposed upon the generating functions of Fourier series and conjugate series, by local conditions we prove the following theorems :

Theorem 1—Let

$$0 < b \leqslant 1,\ b = a(1+b) \text{ and } c(1+b) > 1. \qquad (2\cdot2)$$

Then

$$\phi(t) \log \frac{1}{t} \in BV(0, p), \text{ where } 0 < p \leqslant e^{-2}, \qquad (2\cdot3)$$

implies that $\sum A_n(x) \in \mid R, e(w), 1+b \mid$.

Theorem 2—Let (2·2) hold and let $0 < p < 1$. Then

$$\psi(+0) = 0 \text{ and } \int_0^{p} \log \frac{1}{t} \mid d\psi(t) \mid < \infty \qquad (2\cdot4)$$

imply that $\sum B_n(x) \in \mid R, e(w). 1+b \mid$.

Improving the criterion for the absolute convergence of a Fourier series, due to Mohanty (1950), Chandra (1972*a*) (also see Chandra 1971 and 1972*b*) proved the following :

Theorem C—Let (2·1) hold. Then

$$\{n^{\delta} A(x)\} \in \mid R, \exp(n^{1-\delta}), 1 \mid \; (0 < \delta < 1)$$

implies that $\sum A_n(x) \in \mid C, 0 \mid$.

We observe that the smaller the "δ" the weaker the hypothesis, that is stronger the result. In this paper, we show that if $1 > \delta > (1+b)^{-1}$, where $0 < b \leqslant 1$, non-local condition (2·1) can be replaced by local condition (2·3). Precisely we prove the following :

Theorem 3—Let (2·2) and (2·3) hold. Then, in order that $\sum A_n(x) \in \mid C, 0 \mid$, it is necessary and sufficient that

$$\{n^{1-a} (\log n)^c A_n(x)\} \in \mid R, e(n), 1 \mid. \qquad (2\cdot5)$$

We observe that the greater the "a" the lighter the Tauberian condition and stronger the result.

Besides the above criterion, we also give the following criterion for the conjugate series of the Fourier series:

Theorem 4—Let (2.2) and (2.4) hold. Then, in order that $\Sigma\ B_n(x) \in |\ C, 0|$, it is necessary and sufficient that

$$\{n^{1-a}\ (\log n)^c\ B_n\ (x)\} \quad \in \ |\ R, e(n), 1\ | \ .$$

Remark : The function $(\log x)^c$, involved in Theorems 1, 2, 3 and 4, may be replaced by any one of the following functions: $\log x\ (\log_2 x)^c, \ldots, \log x \ldots \log_{k-1} x\ (\log_k x)^c$, where $\log_1 = \log$ and $\log_k = \log \log_{k-1}$.

3. Order-Estimates

We shall use the following order-estimates in the proofs of the theorems:

Let (2.2) hold. Then, uniformly in $0 < t \leqslant \pi$,

$$\left.\begin{array}{l} E\ (w, t) \\ E\ (w, t) \end{array}\right\} = 0 \left\{ \frac{(\log w)^c\ e^{1+b}\ (w)}{w^a} \right\} \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot1)$$

and

$$\left.\begin{array}{l} E\ (w, t) \\ E\ (w, t) \end{array}\right\} = 0 \left\{ t^{-1}\ w^{-1}\ e^{1+b}\ (w) \right\} . \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot2)$$

For the proof of (3·1) and (3·2) comparision may be made with (3.1.3.) and (3.1.4) of Chandra (1970*b*) .

4. Lemmas

For the proof of the theorems, we require the following lemmas:

Lemma 1 (Obrechkoff 1928, 1929)—Let $r' > r \geqslant 0$. Then $\Sigma\ a_n \in |\ R,\ \lambda_n, r\ |$ implies that $\Sigma\ a_n \in |\ R,\ \lambda_n\ r'\ |$.

Lemma 2—Let the function $F(x)$ and its derivative $F^{(1)}\ (x)$ be positive and monotonic increasing with x for $x > 0$ and let $x > t^{-1}$. Then, uniformly in $0 < t < \eta$, where η is finite and positive number,

$$F\ (x) - F\left(x - \frac{1}{t}\right) \leqslant t^{-1}\ F^{(1)}\ (x); \qquad \ldots \quad \ldots \quad \ldots \quad (4\cdot1)$$

and

$$F\ (x) - F\left(\ (x - \frac{1}{t}\right) \geqslant t^{-1}\ F^{(1)} \left(x - \frac{1}{t}\right) . \quad \ldots \quad \ldots \quad (4\cdot2)$$

PROOF : It may be observed that

$$F(x) - F\left(x - \frac{1}{t}\right) = \int_{x-\frac{1}{t}}^{x} F^{(1)}(y)\ dy .$$

Therefore, since $0 < F^{(1)}(y) \uparrow$ with y for $y > x - \frac{1}{t}$, we have

$$t^{-1} F^{(1)}\left((x - \frac{1}{t}\right) \leqslant \int_{x-\frac{1}{t}}^{x} F^{(1)}(y)\ dy \leqslant t^{-1} F^{(1)}(x)$$

from which we follow the proofs of (4.1) and (4.2).

Lemma 3—We write

and

$$s(x) = \sum_{n \leqslant x}{}' n \cos nt \ ; \quad \bar{s}(x) = \sum_{n \leqslant x}{}' n \sin nt$$

$$s^1(x) = \sum_{n \leqslant x}{}' (x - n)\, n \cos nt \ ; \quad \bar{s}^1(x) = \sum_{n \leqslant x}{}' (x - n)\, n \sin nt.$$

Then uniformly in $0 < t \leqslant \pi$,

$$\left.\begin{matrix} s(x) \\ \bar{s}(x) \end{matrix}\right\} = O(xt^{-1}) \ ; \qquad \ldots \quad \ldots \quad \ldots \qquad (4{\cdot}3)$$

and

$$\begin{matrix} s^1(x) \\ \bar{s}^1(x) \end{matrix} = O(xt^{-2}) . \qquad \ldots \quad \ldots \quad \ldots \qquad (4{\cdot}4)$$

PROOF : The proof of (4.3) immediately follows by Abel's lemma. Therefore, we give the proof of (4.4).

Let $m \leqslant x < m+1$ and let $D_n(t) = \sin(n+\frac{1}{2})\, t/(2 \sin \frac{1}{2} t)$.

Then

$$s^1(x) = \sum_{n=1}^{m}{}' (x - n) n \cos nt$$

$$= \sum_{n=1}^{m-1}{}' \Delta\, ((x - n)n)\ (D_n(t) - \tfrac{1}{2}) + (x - m)m\ (D_m(t) - \tfrac{1}{2})$$

(by Abel's transformation)

$$= \sum_{n=1}^{m-1} (2n + 1 - x)\ D_n(t) + (x - m)m\ D_m(t) + \tfrac{1}{2}(1 - x)$$

$$\leqslant (x + 2m - 1) \max_{1 \leqslant n' \leqslant m} \left| \sum_{n=n'}^{m-1} D_n(t) \right| + m \left| D_m(t) \right| + \tfrac{1}{2}(x-1)$$

(by Abel's Lemma)

$$= O(xt^{-2}) + O(xt^{-1}) + O(x)$$

$$= O(xt^{-2}),$$

uniformly in $0 < t \leqslant \pi$. And, by writing $M_n(t) = \dfrac{\cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t}$ and proceeding as above, we obtain that

$$\bar{s}^1(x) = \sum_{n=1}^{m-1} \Delta((x-n)n)\ (\tfrac{1}{2}\cot \tfrac{1}{2}t - M_n(t)) + (x-m)m\ (\tfrac{1}{2}\cot \tfrac{1}{2}t - M_m(t))$$

$$= \tfrac{1}{2}(x - 1)\cot \tfrac{1}{2}t - (x - m)\ m\ M_m(t) - \sum_{n=1}^{-1} (2n + 1 - x)\ M_n(t)$$

$$= O(xt^{-1}) + O(xt^{-2})$$

$$= O(xt^{-2})$$

proceeding as above, uniformly in $0 < t \leqslant \pi$.

This completes the proof of the Lemma.

Lemma 4—Uniformly in $0 < t \leqslant \pi$ and for $w > t^{-1}$,

$$\left.\begin{array}{l} K(w, t) \\ \bar{K}(w, t) \end{array}\right\} = O\left\{ t^{-2} e^{1+b}(w) \frac{w^{a-1}}{(\log w)^c} \right\} + O\left\{ t^{-1-b} e^{1+b}(w) \frac{w^{b(a-1)}}{(\log w)^{bc}} \right\}.$$

Proof : By using the notations of Lemma 3, we obtain that

$$\left.\begin{array}{l} K(w, t) \\ \bar{K}(w, t) \end{array}\right\} = -\int_1^w \left\{\begin{array}{l} s(x) \\ \bar{s}(x) \end{array}\right\} \cdot \frac{d}{dx}\left\{ (e(w) - e(x))^b \frac{e(x)}{x} \right\} dx$$

$$= -\int_1^w \left\{\begin{array}{l} s(x) \\ \bar{s}(x) \end{array}\right\} \cdot (e(w) - e(x))^b \frac{e^{(1)} x}{x}\ dx$$

$$+ \int_1^w \left\{\begin{array}{l} s(x) \\ \bar{s}(x) \end{array}\right\} \cdot (e(w) - e(x)^b \frac{e(x)}{x^2}\ dx$$

$$+ \; b \int_1^w \left. \begin{matrix} s(x) \\ \bar{s}(x) \end{matrix} \right\} \cdot \frac{e(x)}{x} \, (e(w) - e(x))^{b-1} \, e^{(1)}(x) \, dx$$

$$= - I_1 + I_2 + bI_3, \text{ say.}$$

Since $x^{-1} \, e^{(1)}(x)$ is positive and increasing with x from some point on, say x_0, we obtain, by using the second mean value theorem two times, that

$$I_1 = O\{e^b(w)\} + \frac{e^{(1)}(w)}{w} \, (e(w) - e(x_0))^b . \max_{\substack{x_0 < \eta < \eta' < w \\ x_0 < \xi < \xi' < w}} \left\{ \left| \int_\eta^{\eta'} s(x) \, dx \right| , \left| \int_\xi^{\xi'} \bar{s}(x) \, dx \right| \right\}$$

$$= O\{e^b(w)\} + O\left\{ \frac{e^{(1)}(w)}{w} e^b(w) \, wt^{-2} \right\}$$

[Lemma 3, (4·4)]

$$= O\left\{ t^{-2} \, e^{1+b}(w) \, \frac{w^{a-1}}{(\log w)^c} \right\},$$

uniformly in $0 < t \leqslant \pi$. Similarly, proceeding as in I_1, it may be obtained that

$$I_2 = O\{ t^{-2} \, e^{1+b}(w) \, w^{-1} \}$$

uniformly in $0 < t \leqslant \pi$. Now, writing w_1 for $\left(w - \frac{1}{t} \right)$, we write

$$I_3 = \int_1^{w_1} + \int_{w_1}^{w} = I_{3,1} + I_{3,2} \text{ , say.}$$

Proceeding as in I_1, we obtain that

$$I_{3,1} = O\{e^b(w)\} + O\{t^{-2} \, e(w) \, (e(w) - e(w_1))^{b-1} \, e^{(1)}(w_1)\}$$

$$= O\{e^b(w)\} + O\{t^{-2} \, e(w) \, (t^{-1} \, e^{(1)}(w_1))^{b-1} \, e^{(1)}(w_1)\}$$

[by Lemma 2, (4·2)]

$$= O\{t^{-1-b} \, e(w) \, (e^{(1)}(w))^b\}$$

$$= O\left\{ t^{-1-b} \, e^{1+b}(w) \, \frac{w^{b(a-1)}}{(\log w)^{bc}} \right\},$$

uniformly in $0 < t \leqslant \pi$. And, by Lemma 3; (4.3), we obtain that

$$J_{3,2} = O\ \{t^{-1} \int_{w_1}^{w} e(x)(e(w) - e(x))^{b-1} e^{(1)}(x)\ dx\}$$

$$= O\ \{t^{-1}\ e(w)\ (e(w) - e(w_1))^{b}\}$$

$$= O\ \{t^{-1-b}\ e(w)\ (e^{(1)}(w))^{b}\}$$

[by Lemma 2, (4·1)]

$$= O\ \left\{ t^{-1-b}\ e^{1+b}\ (w)\ \frac{w^{b(a-1)}}{(\log w)^{bc}} \right\} ,$$

uniformly in $0 < t \leqslant \pi$.

Lemma 5 (Das 1969, *Lemma* 17)—For any sequence $\{L_n\}$, $\{b_n\} \in | R', L_n, 1 |$ and $\{d_n\} \in BV$ imply $\{b_n d_n\} \in | R', L_n, 1 |$.

This is explicitly proved by Das for $| R', L_{n-1}, 1 |$ and by the same arguments, the above result holds good for $| R', L_n, 1 |$ which is equivalent to $| R, L_n, 1 |$ (see Mohanty 1951, footnote to the page 298).

Lemma 6—Let $\Sigma a_n \in | R, \exp\{n^a/(\log n)^c\}, r |$, where $0 < a < 1$, $c > 0$ and $r > 0$. Then in order that $\Sigma a \in | C, 0 |$. it is necessary and sufficient that $\{n^{1-a} (\log n)^c a_n\} \in | R, \exp\{n^a/(\log n)^c\}, 1 |$ .. .. .. (4·5)

PROOF : *Condition is necessary*—Let $s_n = a_1 + a_2 + \ldots + a_n$ and $\{s_n\} \in BV$. Then, for any sequence $\{L_n\}$, $\{s_n\} \in | R', L_n, 1 |$,

which is equivalent to

$$\left\{ s_n - \frac{1}{L^{n+1}} \sum_{m=1}^{n} L_m a_m \right\} \in BV ,$$

and, which further implies that

$$\left\{ \frac{a_n L_n}{L_{n+1} - L_n} \right\} \in | R', L_n, 1 | . \quad .. \quad .. \quad .. \quad (4·6)$$

Since, for $L_n = \exp\{n^a/(\log n)^c\}$ $(0 < a < 1$ and $c > 0)$,

$\{n^{1-a} (L_{n+1} - L_n) (\log n)^c / L_n\} \in BV$, ... (4·7)

therefore, by using (4·6) and (4·7) in Lemma 5, we follow the proof of the necessity part.

Condition is sufficient—Since for L_n , as defined above,

$$\left\{ \frac{n^{a-1} L_n}{(L_{n+1} - L_n) (\log n)^c} \right\} \in BV$$

therefore (4·5), by Lemma 5, implies (4·6), which further implies and is implied by

$$\text{(i)} \left\{\frac{L_n}{L_{n+1}}\right\} \in BV \text{ and (ii) } \left\{\frac{1}{L_n}\sum_{m=1}^{n} L_m a_m\right\} \in BV.$$

Therefore (4·5) is sufficient since (i) and (ii) are sufficient conditions (see Bhatt 1958) for $\Sigma a_n \in | C, 0 |$, whenever $\Sigma a_n \in | R, L_n, r | \ (r > 0)$.

This terminates the proof of the lemma.

Lemma 7—The integral

$$I = \int_0^\infty \frac{e^{(1)}(w)}{e^{2+b}(w)} \, dw \left| \int_0^t \frac{K(w, u)}{\log \frac{k}{u}} \, du \right|$$

$$= O(1),$$

uniformly in $0 < t \leqslant \pi$, where $k \geqslant \pi e^2$.

PROOF : Integrating by parts, we have

$$\int_0^t \frac{K(w, u)}{\log \frac{k}{u}} \, du = \frac{E(w, t)}{\log \frac{k}{t}} - \int_0^t \frac{E(w, u)}{u \left(\log \frac{k}{u}\right)^2} \, du .$$

Therefore

$$I \leqslant \left(\log \frac{k}{t} \right)^{-1} \int_e^\infty \frac{e^{(1)}(w)}{e^{2+b}(w)} \, | E(w, t) | \, dw$$

$$+ \int_e^\infty \frac{e^{(1)}(w)}{e^{2+b}(w)} \left| \sum_{n \leqslant w} (e(w) - e(n))^b \, e(n) \, \alpha_n \right| dw$$

$$= I_1 + I_2, \text{ say},$$

where

$$\alpha_n = \frac{1}{n} \int_0^t \frac{\sin nu}{u \left(\log \frac{k}{u}\right)^2} \, du$$

$$= O\left\{ \frac{1}{n (\log n)^2} \right\},$$

by moving parallel to (3·2) of Chandra (1970*a*)*. Thus the convergence of I_2 follows from Lemma 1.

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*Read $g\left(\frac{1}{t}\right)$ for $g\left(\frac{k}{t}\right)$ throughout the paper of Chandra (1970*a*) and also (2.4) of this paper should be read for $x > 0$.

Now, for $T = \left(\frac{k}{t}\right)^{1/(1-a)}$, we write

$$I_1 = \int_0^T + \int_T^\infty = I_{1,1} + I_{1,2}, \quad \text{say.}$$

The boundedness of $I_{1,i}$ $(i = 1,2)$, uniformly in $0 < t \leqslant \pi$, follow by using (3·1) and (3·2) for $E(w,t)$ respectively.

This completes the proof of the lemma.

5. Proof of the Theorems

We shall first prove Theorems 1 and 2 and then by using them we shall prove Theorems 3 and 4.

Proof of Theorem 1—Since

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \quad dt ,$$

the series $\Sigma A_n(x) \in | R, e(w), 1+b|$ $(1 \geqslant b > 0)$, if (see Mohanty 1951, definition B)

$$I = \frac{2}{\pi} \int_0^\infty \frac{e^{(1)}(w)}{e^{2+b}(w)} \left| \int_0^\pi \phi(t)\, K(w, t)\, dt \right| dw .$$

Now, for $0 < p \leqslant e^{-2}$, we have

$$I \leqslant \frac{2}{\pi} \int_e^\infty \frac{e^{(1)}(w)}{e^{2+b}(w)} \left\{ \left| \int_0^p \phi(t) K(w, t)\, dt \right| + \left| \int_p^\pi \phi(t) K(w, t)\, dt \right| \right\} dw$$

$$= I_1 + I_2, \quad \text{say.}$$

Since the boundedness of I_1 follows from Lemma 7, the boundedness of I_2 will be sufficient for the proof of Theorem 1. And for I_2, we have, by using Lemma 4 for $K(w, t)$, that

$$I_2 = O\left\{ \int_p^\pi t^{-2} \mid \phi(t) \mid \, \mathrm{dt} \int_e^\infty \frac{w^{2(a-1)}}{(\log w)^{2c}}\, dw \right\}$$

$$+ O\left\{ \int_p^\pi t^{-1-b} \mid \phi(t) \mid dt \int_e^\infty \frac{w^{(1+b)(a-1)}}{(\log w)^{c(1+b)}}\, dw \right\}$$

$$= O\left\{\int_p^\pi |\phi(t)|\, dt \int_e^\infty \frac{1}{w(\log w)^{c(1+b)}}\, dw\right\} \qquad \text{(b y(2.2))}$$

$$= O\left\{\int_p^\pi |\phi(t)|\, dt\right\}$$

(by (2·2))

$$= O\ (1).$$

This terminates the proof of Theorem 1.

Proof of Theorem 2—We have

$$B_n(x) = \frac{2}{\pi}\int_p^\pi \psi(t)\sin nt\, dt\ .$$

Integrating by parts and taking $0 < p < 1$, we have

$$B_n(x) = -2\psi(p)\frac{\cos np}{n\pi} + \frac{2}{\pi}\int_p^\pi \frac{\cos nt}{n}\, d\psi(t)$$

$$+ \frac{2}{\pi}\int_p^\pi \psi(t)\sin nt\, dt\ .$$

Proceeding as in Theorem 1, it is sufficient to show that

$$I_1 = \int_e^\infty \frac{e^{(1)}(\omega)}{e^{2+b}(\omega)}\ |\bar{E}(\omega, t)|\ dw = O\left(\left(\log \frac{k}{t}\right)\right),$$

uniformly in $0 < t \leqslant p$. And

$$I_2 = \int_p^\pi |\psi(t)|\ dt \int_e^\infty \frac{e^{(1)}(\omega)}{e^{2+b}(\omega)}\ |K(w, t)|\ dw < \infty\ .$$

The convergence of I_2 follows by making use of Lemma 4 for $\bar{K}(w, t)$. Therefore, for the proof of the theorem, it remains to show that

$$I_1 = O\left(\log \frac{k}{t}\right),$$

uniformly in $0 < t \leqslant p$.

Now, for $T = \left(\dfrac{k}{t}\right)^{1/(1-a)}$, we write

$$I_1 = \int_e^T + \int_T^\infty = I_{1,1} + I_{1,2}, \text{ say.}$$

By (3·1), for $\bar{E}(w, t)$, we have

$$I_{1,1} = O\left\{\int_e^T w^{-1}\, dw\right\} = O\left(\log \frac{k}{t}\right),$$

uniformly in $0 < t \leqslant p$. And finally by (3·2) for $\bar{E}(w, t)$, we have

$$\begin{aligned} I_{1,2} &= O\left\{t^{-1}\int_T^\infty \frac{w^{a-2}}{(\log w)^c}\, dw\right\} \\ &= O\left\{\frac{T^{a-1}}{t(\log \frac{k}{t})^c}\right\} \\ &= O\left\{(\log \tfrac{k}{t})^{-c}\right\}, \end{aligned}$$

uniformly in $0 < t \leqslant p$. Combining $I_{1,1}$ and $I_{1,2}$, we follow the proof.

This completes the proof of the theorem.

Proof of Theorems 3 *and* 4—For the proofs of these theorems we use the standard definition of absolute Riesz summability. We first sketch the proof of Theorem 3.

By Theorem 1, we follow that $\Sigma A_n(x) \in |R, e(n), 1+b|$, whenever (2·2) and (2·3) hold. And, therefore, by Lemma 6, $\Sigma A_n(x) \in |C, 0|$ if and only if (4·5), with $A_n(x)$ in place of a_n, holds. Again by using $|R', L_n, 1| \sim |R, L_n, 1|$ (see remark to Lemma 5) we follow the proof of Theorem 3.

For the proof of Theorem 4, we proceed as above and use Theorem 2 in place of Theorem 1.

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