A NOTE ON SELF COMMUTATORS

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In the present note it has been established that a self adjoint operator (on a separable Hilbert space) whose ascent and descent are not equal is a self commutator. The converse of this result is not true. An alternative proof of a theorem by Halmos has also been given.

The study of self commutators on Hilbert space was initiated by Halmos (1952). Radjavi (1966) obtained the following characterization of self commutators on a separable Hilbert space:

Let K be any self adjoint operator on an infinite dimensional separable Hilbert space H. Then the following statements are equivalent:

(i) There exists an operator A on H such that $A^*A—AA^* = K$.

(ii) There exist two self adjoint operators M and N such that $MN—NM=iK$.

(iii) There exists an infinite orthonormal sequence $<e_j>$ of vectors in H such that $\left| \sum_{j=1}^{n} (Ke_j, e_j) \right|$ is bounded Vn.

(iv) $Sp\ (K)$ has at least one non-negative and one non-positive limit point.

(v) K is not of the form $K_1 \oplus K_2$, where K_1 has finite-dimensional domain and K_2 satisfies either $K_2 \geqslant \epsilon > 0$ or $Q_2 \leqslant \epsilon < 0$ for some real number ϵ. Using this characterization the author has obtained many properties of self commutators (Taylor 1958). Our object in the present note is to obtain certain sufficient conditions for a self adjoint operator to be a self commutator. Throughout our work, H denotes an infinite dimensional separable (complex) Hilbert space and $<f_n>_{n=1}^{\infty}$ = an orthonormal basis of it. All operators considered are bounded and defined ont he whole of H. $N(K)$, $R(K)$, $\mathscr{D}(K)$ denote respectively the null space, the range space and the domain space of the operator K. For the definitions of the ascent $\alpha\ (K)$ and the descent $\delta\ (K)$ of K we refer to Taylor (1958 or 1966).

Theorem 1—If K is a self adjoint operator with unequal ascent and descent, then K is a self commutator.

For the proof we require the following:

Lemma—The following statements are equivalent for a self adjoint operator K on H.

(S): K is of the form $K_1 \oplus K_2$, where K_1 has finite-dimensional domain and K_2 satisfies either $K_2 \geqslant \epsilon > 0$ or $K_2 \leqslant \epsilon < 0$ for some real number ϵ.

(S_2) : K is of the form $C \pm P$, where

(a) C is a compact operator.

(b) $P \geqslant \epsilon > 0$ for some real number ϵ,

(c) $CP = PC$

Proof of the Lemma : Let K satisfy (S_1) with $K_2 \geqslant \epsilon > 0$. Write

$$K = ((K_1 - \epsilon) \oplus 0) + (\epsilon \oplus K_2),$$
$$= C + P.$$

Then C and P satisfy (a), (b) and (c) in (S_2). Conversely if K satisfies (S_2), then $P \geqslant \epsilon > 0$ and hence $Sp(P)$ has at least one non-negative and one non positive limit point is not satisfied. Since perturbation by a compact operator leaves the set of limit points invariant, hence K does not satisfy (iv) of Radjavi's (1966) characterization and therefore satisfies (S_1).

Proof of the Theorem: Suppose K is not a self commutator. Then K is of the form $C+P$ (if it is of the form $C-P$ we can work with $-K$), where C and P satisfy (a), (b) and (c) of the lemma. Since for a self adjoint operator K, $K^n x=0$ if and only if $K^{n+1} x=0$, therefore $N(K) = N(K^2)$. Hence ascent $\alpha(K)$ of K is finite and is 0 or 1. Assume that the descent $\delta(K)$ of K is not finite. Then

$$R(K^0) \underset{\neq}{\supset} R(K) \underset{\neq}{\supset} R(K^2) \underset{\neq}{\supset} \ldots . \underset{\neq}{\supset} R(K^{n-1}) \underset{\neq}{\supset} R(K\ ) \ldots .$$

Now $K^n =$ compact operator $+ P^n$, and hence K^n is invertible modulo the ideal of compact operators, therefore by Halmos (1967, Problem 142), $R(K^n)$ is closed $\forall\ n$. Hence $R(K^{n+1})$ is a proper closed subspace of $R(K^n)$, $n=0, 1,2. \ldots$ and therefore we can choose $Y_n \in R(K^n)$ with $\| y_n \| = 1$ and satisfying $\| y_n - y \| \geqslant \frac{1}{2}\ \forall\ y \in R(K^{n+1})$. Let $1 \leqslant m < n$. Put

$$Z = y_n - P^{-1} K y_n + P^{-1} K\, y_m .$$

$y_n \in (R\ (K^n)$ implies that there exists x_n such that $y_n = K^n X_n$ and hence

$$Z = K^n x_n - P^{-1} K^{n+1} x_n + P^{-1} K^{m+1} x_m$$
$$= K^{m+1} u,$$

for some u. Therefore $Z \in R\ (K^{m+1}\ )$ and hence $\| y_m - Z \| \geqslant \frac{1}{2}$. This gives

$$Cy_n - Cy_m = P(y_m - Z).$$

Hence

$$\| Cy_n - Cy_m \|^2 \geqslant a^2 \| y_m - z \|^2 \geqslant \frac{a^2}{4}.$$

Therefore

$$\| Cy_m - Cy_n \| \geqslant a/2 > 0$$

This is a contradiction because $< y_n >$ is a bounded sequence and C is a compact operator. Hence $\delta(K)$ is also finite. Since $\mathscr{D}(K) = H$, therefore $\alpha(K) = \delta(K)$ Taylor 1958, Theorem 3.6). A contradiction implies that K is a self commutator.

Consider an operator K on H as

$$Kf_1 = f_1,$$
$$Kf_2 = f_2,$$

and

$$Kf_n = 0 \quad \forall n > 2,$$

then K is a self commutator (Halmos 1952, Lemma 2). But $\alpha(K) = \delta(K)$ 1.

We now give an alternative proof of the following due to Halmos (1952, Lemma 2).

Theorem 2—A self adjoint operator with an infinite-dimensional null space is a self commutator.

PROOF: Suppose K is not so. Then $K = C+P$ where C and P satisfy (a), (b) and (c) of the Lemma. Put $S = \{x : \| x \| = 1\} \cap N(K)$. Let $< x_n >$ be any sequence in S. Then $\| x_n \| = 1$ and $Kx_n = 0$. This gives $x_n = -P^{-1}Cx_n$. But $P^{-1}C$ is a compact operator and $\| x_n \| = 1 \ \forall n$, therefore x_n has a convergent subsequence. Hence S is a compact subset of $N(K)$ which by Theorem 3·12 F of Taylor (1958) implies that $N(K)$ is a finite-dimensional space. A contradiction implies that K should be a self commutator.

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