EMBEDDING CLASS OF A CONFORMALLY FLAT PERFECT FLUID DISTRIBUTION

by Y. K Gupta, *Department of Mathematics, University of Roorkee, Roorkee*

(Communicated by F.C. Auluck, F.N.A.)

(Received 25 July 1974)

If a conformally flat space-time describes a perfect fluid distribution of matter, (with $\rho \neq 0$) then it is necessarily of embedding class one and the lines of flow are normal to the hypersurfaces $\rho(x^i) =$ constant. Since conformally flat electromagnetic fields of class one do not exist, it follows that the only physically significant situation described by a conformallyflat space of class one is a perfect fluid distribution.

1. Introduction

Pandey and Gupta (1969) have shown that class one and perfect fluid conditions are identical in the case of a spherically symmetric and conformally flat space-time. The same result has been proved by Rao (1971) afterwards with an alternative approach. In the current paper the result is established in general. Moreover the lines of flow are found normal to the hypersurface. $\rho(x^i) =$ const. when the density of the distribution is not constant everywhere.

If a space-time is conformal to a flat space-time, there exists a coordinate system in which its first fundamental form can be expressed as

$$ds^2 = \psi\, [(dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2] \qquad \ldots \qquad \ldots \qquad (1\cdot1)$$

where $\psi = \psi\ (x^4, x^1, x^2, x^3)$.

It is well know that (1·1) is at the most of class two (Eisenhart 1960). Now it will be shown that the space-time expressed by (1·1) turns out to be of class one, if it describes a perfect fluid distribution.

2. Conformally flat space-time and perfect fluid distribution

In case of a conformally flat space-time, the curvature tensor can be expressed as

$$R_{hijk} = \tfrac{1}{2}\, [- g_{hj}\, L_{ik} + g_{hk}\, L_{ij} - g_{ik}\, L_{hj} + g_{ij}\, L_{hk}] \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2\cdot1)$$

where

$$L_{ij} = R_{ij} - \tfrac{1}{6}\, g_{ij} R. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2\cdot2)$$

For a perfect-fluid distribution of matter

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -8\pi \left[(p + \rho) v_i v_j - p g_{ij} \right] \quad \ldots \quad (2\cdot3)$$

$$g^{ij} v_i v_j = 1 \quad \ldots \quad (2\cdot4)$$

where p, ρ and v^i are pressure, density and flow-vector characterizing the fluid distribution.

In consequence of (2·3) and (2·4), L_{ij} in (2·2) assumes the form

$$L_{ij} = -8\pi \left[(p + \rho) v_i v_j - \tfrac{1}{3} \rho g_{ij} \right] \quad \ldots \quad (2\cdot5)$$

When this expression for L_{ij} is inserted in (2·1), we get

$$R_{hijk} = 4\pi (p + \rho) \left[g_{hj} v_i v_k - g_{hk} v_i v_j + g_{ik} v_h v_j - g_{ij} v_h v_k \right] - \frac{8\pi\rho}{3} (g_{hj} g_{ik} - g_{hk} g_{ij}). \quad \ldots \quad (2\cdot6)$$

On the other hand for a Riemannian fourfold of class one, the curvature tensor is expressible as

$$R_{hijk} = e (b_{hj} b_{ik} - b_{hk} b_{ij}), \quad e = \pm 1 \quad \ldots \quad (2\cdot7)$$

while

$$b_{ij;k} - b_{ik;j} = 0 \quad \ldots \quad (2\cdot8)$$

where (;) denotes covariant differentiation.

It will be shown that b_{ij}, satisfying (2·7) and (2·8) exists, if (2·6) is true. In case of a conformally flat perfect fluid of class one, one can take b_{ij} in the form (Pandey and Gupta 1970).

$$b_{ij} = \lambda v_i v_j + \mu g_{ij} \quad \ldots \quad (2\cdot9)$$

where λ and μ are scalars. In consequence of (2·9), the curvature tensor in (2·7) assumes the form

$$R_{hijk} = e \lambda\mu \left[g_{hj} v_i v_k - g_{hk} v_i v_j + g_{ik} v_k v_j - g_{ij} v_h v_k \right] + e \mu^2 (g_{hj} g_{ik} - g_{hk} g_{ij}) \quad \ldots \quad (2\cdot10)$$

Comparing (2·6) and (2·10), we find

$$e \mu^2 = -\frac{8\pi}{3} \rho \quad \ldots \quad (2\cdot11)$$

and $e\mu \lambda = 4\pi (p + \rho)$, $\quad \ldots \quad (2\cdot12)$

which demands that $e = -1$ for a positive density.

In consequence of (2·9), (2·11) and (2·12), it follows that

$$b_{ij} = \pm \left[\sqrt{\frac{6\pi}{\rho}} (p + \rho) v_i v_j - \sqrt{\frac{8\pi \rho}{3}} \cdot g_{ij} \right], \text{ provided } \rho \neq 0 \quad \ldots \quad (2\cdot13)$$

Thus we have proved that (2·6) can be expressed as (2·7) with b_{ij} given by (2·13). Hence a perfect fluid distribution of matter, described by a conformally flat metric, is of class one, provided that (2·13) satisfy (2·8). Thomas (1963) has proved that the equation (2·8) is a consequence of (2·7), if the determinant of b_{ij} is non-zero. We shall now prove that $| b_{ji} | \neq 0$ unless $\rho + 3p = 0$. However, in the later case also it will be shown that (2·8) is satisfied identically provided (2·7) holds good.

In view of (1·1) and (2·9), it can be proved that

$$| b_{ij} | = \mu^3 \lambda \psi^3 (v_1^2 + v_2^2 + v_3^2 - v_4^2) - \mu^4 \psi^4 \quad \ldots \quad \ldots \quad \ldots \quad (2·14)$$

From (2·4) and (1·1), we get

$$v_1^2 + v_2^2 + v_3^2 - v_4^2 = -\psi. \quad \ldots \quad \ldots \quad \ldots \quad (2·15)$$

When (2·15) is inserted in (2·14), we have

$$| b_{ij} | = -\mu^3 \psi^4 (\lambda + \mu). \quad \ldots \quad \ldots \quad \ldots \quad (2·16)$$

We see from (2·10) that μ can not be zero, otherwise the space-time will turn out to be flat. Therefore,

$$| b_{ij} | \neq 0 \text{ unless } \lambda + \mu = 0$$

which is the same thing as $3p + \rho = 0$ as it can be verified from (2·11) and (2·12).

In the later case when $\lambda + \mu = 0$, (2·9) can be expressed as

$$b_{ij} = \mu (- v_i v_j + g_{ij}). \quad \ldots \quad \ldots \quad \ldots \quad (2·17)$$

The Bianchi-identities with reference to (2·7) give rise

$$b_{ik} B_{hjl} + b_{hj} B_{ikl} + b_{il} B_{hkj} + b_{hk} B_{ilj} + b_{ij} B_{hlk} + b_{hl} B_{ijk} = 0 \quad \ldots \quad \ldots \quad (2·18)$$

where $B_{hjl} \equiv b_{hj;l} - b_{hl;j}$. .. .. .. (2·19)

When (2·17) is inserted into (2·19) we get

$$B_{hjl} = \mu_{,l} (- v_h v_j + g_{hj}) - \mu (v_{h;l} v_j + v_h v_{j;l})$$
$$- \mu_{,j} (- v_h v_l + g_{hl}) + \mu (v_{h;j} v_l + v_h v_{l;j}). \quad \ldots \quad \ldots \quad (2·20)$$

(2·18) on contraction with $g^{hj} g^{ik} v^l$ and $v^h g^{ik} v^l$ and using (2·20) gives

$$3\mu_{,k} v^k + \mu v^k_{;k} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2·21)$$

and

$$v_{i;k} - v_{k;i} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2·22)$$

respectively.

Again contracting (2·18) by $g^k v^l$ and using (2·20), (2·21) and (2·22) we get,

$$v^l B_{ikl} = \mu v_{i,k} - \mu_j v^j (v_i v_k - g_{ik}) = 0. \quad \ldots \quad \ldots \quad \ldots \quad (2·23)$$

Now contracting again (2·18) with $g^{il}\, g^{hj}$ and using (2·20), (2·21), (2·22) and (2·23) we get

$$\mu_{,k} = \mu_{,j}\, v^j\, v_k. \qquad (2\cdot24)$$

Owing to the results (2·22), (2.23) and (2·24) we arrive

$$B_{hjl} = 0$$

and hence the proposition.

However when $\mu_{,k} = 0$ i.e. ρ = const. every where, working upto the step (2·23) will serve the purpose.

3. Lines of Flow and Hypersurfaces $\rho(x^i)$ = const.

In this section it is established that the lines of flow are normal to the hypersurface ρ = const. in case of a conformally flat perfect fluid (obviously of class one) so that

$$v_i = \phi\, \rho_{,i}\,, \qquad (3\cdot1)$$

where, ϕ is a scalar.

Contracting (2·8) by g^{lj}, $g^{lj}\, v^k$ and $v^l\, v^j$ and using (2·9) we get three relations

$$\lambda_{,k} + 3\mu_{,k} - \lambda_{,j}\, v^j v_k - \lambda(v^j_{;j}\, v_k + v^j\, v_{k;j}) = 0, \qquad (3\cdot2)$$

$$3\mu_k\, v^k - \lambda v^j_{;j} = 0, \qquad (3\cdot3)$$

and

$$\lambda_{,k} + \mu_{,k} - \lambda_{,j}\, v^j v_k - \mu_{,j}\, v^j\, v_k - \lambda v^j\, v_{k,j} = 0. \qquad (3\cdot4)$$

Using (3·3) and (3·4) in (3·2) we get

$$\mu_{,k} = \mu_{,j}\, v^j v_k \qquad (3\cdot5)$$

(3·5) and (2·11) implies (3·1) and hence the result.

Since on electromagnetic field which is conformal to a flat space and also of class one does not exist (Collinson 1968) the only physically significant distribution described by a conformally flat space-time is a perfect fluid distribution, which is always of class one having lines of flow normal to the hypersurvaces $\rho(x^i)$ = const.

Acknowledgement

The author wishes to put on record his thanks to Dr. S. N. Pandey, Reader in Mathematics, University of Roorkee, for valuable discussions regarding the problem.

References

Collinson, C. D. (1968). Einstein-Maxwell fields of embedding class one. *Commun. Math. Phys.*, **8**, 1-11.

Eisenhart, L. P. (1960). Riemannian Geometry, p. 215.

Pandey, S. N., and Gupta. Y.K. (1969). Spherically symmetric conformally flat space-time of class one. *Univ. Roorkee Res. J.*, XI, 7-11.

———(1970). Space-time of class one and perfect fluid distribution. *Univ. Roorkee Res. J.*, XII, 1-11.

Rao, J. Krishna (1971). On spherically symmetric perfect fluid distributions and class one property. *General Relativity and Gravitation,* **2,** No. 4., 385-86.

Thomas, T. Y. (1963). Riemannian spaces of class one and their characterization. *Acta. Math.*, **67,** 169-211.