# THE COMPLEX LAPLACE-HANKEL TRANSFORMATION OF GENERALIZED FUNCTIONS

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In this paper the testing function space $B_{\lambda, a, b}$ has been constructed so that the kernel $e^{-ux}\sqrt{vy}\, J_\lambda(vy)$ belongs to $B_{\lambda, a, b}$ for some restricted $(u, v)$. The complex Laplace-Hankel transform $F(u, v)$ of a generalized function $f$ on $B_{\lambda, a, b}$ is defined directly as the application of $f$ to $e^{-ux}\sqrt{vy}\, J_\lambda(vy)$ i.e.

$$F(u, v) = < f(x, y),\ e^{-ux}\sqrt{vy}\, J_\lambda(vy) >$$

for some $(u, v)$. The inversion formula for the generalized complex Laplace-Hankel transformation has been also derived.

## 1. Introduction

The authors (*in priss*) have extended the Laplace-Hankel transformation to certain class of generalized functions by generalizing the Parselval's equation as follows: For any real numbers $a$ and $\lambda > -\frac{1}{2}$ the spaces $\boldsymbol{LH}_{a,\lambda}$, $\check{\boldsymbol{LH}}_{a,\lambda}$ and $\tilde{\check{\boldsymbol{LH}}}_{a,\lambda}$ have been constructed such that the mapping $\phi = LH_\lambda(\phi) \rightarrow 2\pi i\, \check{\phi}$ is an isomorphism from $\tilde{\check{\boldsymbol{LH}}}_{a,\lambda}$ onto $\boldsymbol{LH}_{a,\lambda}$, where $LH_\lambda$ denotes the conventional Laplace-Hankel transformation and $\check{\phi}(x, y) = \phi(-x, y)$, $\phi \in \check{\boldsymbol{LH}}_{a,\lambda}$ and $\check{\phi} \in \boldsymbol{LH}_{a,\lambda}$. For any generalized function $f \in \boldsymbol{LH}'_{a,\lambda}$ the Laplace-Hankel transform $LH_\lambda'(f)$ was defined by

$$< LH_\lambda'(f), LH_\lambda(\phi) > = 2\pi i < f, \check{\phi} > \qquad \cdots \qquad (1\cdot1)$$

where $\boldsymbol{LH}_\lambda(\phi) \in \tilde{\check{LH}}_{a,\lambda}$ and $\check{\phi} \in \boldsymbol{LH}_{a,\lambda}$. The generalized Laplace-Hankel transformation $\tilde{\check{LH}}_{a,\lambda}$ is an isomorphism from $\boldsymbol{LH}'_{a,\lambda}$ onto $\tilde{\check{LH}}'_{a,\lambda}$, where $\boldsymbol{LH}'_{a,\lambda}$ and $\tilde{\check{\boldsymbol{LH}}}'_{a,\lambda}$ denote the dual spaces of $\boldsymbol{LH}_{a,\lambda}$ and $\check{\boldsymbol{LH}}_{a,\lambda}$ respectively.

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In constrast to this, in the present work, we define the Laplace-Hankel transform $F(u, v)$ of a generalized function $f$ directly as the application of $f(x,y)$ to $e^{-ux} \sqrt{vy}\ \mathcal{J}_\lambda(vy)$

i.e.

$$F(u, v) = < f(x,y), e^{-ux} \sqrt{vy}\ \mathcal{J}_\lambda (vy) > . \qquad (1{\cdot}2)$$

For this purpose, we have constructed the testing function space $B_{\lambda, a, b}$ on $0 < x < \infty$ , $0 < y < \infty$ which contains the kernel $e^{-ux} \sqrt{vy}\ \mathcal{J}_\lambda (vy)$ for all $(u, v)$ in some restricted domain.

In this paper, we shall also extend the classical inversion formula for generalized functions in $B'_{\lambda, a, b}$ .

The notation and terminology of this work follows that of Zemanian (1968) and Koh and Zemanian (1968). $I$ denotes the open set $0 < x < \infty$ , $0 < y < \infty$ . $D(I)$ is the space of infinitely differentiable functions defined on $I$ having compact support in $I$. $D'(I)$ is the dual space with its customary topology [Zamanian 1968, p. 33] $D_x^k$ and $S_{\lambda, y}^{k'}$ denote respectively $\frac{\partial^k}{\partial x^k}$ and

$$\left( y^{-\lambda-\frac{1}{2}} \frac{\partial}{\partial y} y^{2\lambda+1} \frac{\partial}{\partial y} y^{-\lambda-\frac{1}{2}} \right)^{k'}$$ where $k$ and $k'$ are non-negative integers in $R'$.

## 2. Testing Function Spaces $B_{\lambda, a, b}$, $B(\lambda, w, z)$ and Their Duals

For any fixed real numbers $a$ and $\lambda$ and fixed positive real number $b$, an infinitely differentiable complex-valued function $\phi(x, y)$ defined on $I$ belongs to $B_{\lambda, a, b}$ if

$$\beta^\lambda_{a, b, k, k'} (\phi) = \sup_{\substack{0<x<\infty \\ 0<y<\infty}} | e^{ax-by} y^{-\lambda-\frac{1}{2}} D_x^k S_{\lambda, y}^{k'} \phi (x, y) | < \infty \qquad (2{\cdot}1)$$

for all $k, k' = 0, 1, 2, \ldots\ldots$

Clearly, $B_{\lambda, a, b}$ is a vector space over the field of complex numbers. $\phi(x, y) = 0$ on $I$ is the zero element of $B_{\lambda, a, b}$ . Since $\beta^\lambda_{a, b, 0, 0}$ is norm, the countable collection of seminorms $\{\beta^\lambda_{a, b, k, k'}\}_{k, k=0}^{\infty}$ is a countable multinorm for $\beta_{\lambda, a, b}$ (Zamanian 1968, p.8). The topology of $B_{\lambda, a, b}$ is generated by the multinorm $\{\beta^\lambda_{a, b, k, k'}\}$ (Zamanian 1968, p.9). Hence, thereby it becomes a countably multinormed space. We say that a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ where $\phi_\nu \in B_{\lambda, a, b}$ converges $B_{\lambda, a, b}$ to $\phi$ if for each pair of non-negative integers $k$ and $k'$, $\beta^\lambda_{a, b, k, k} (\phi_\nu - \phi) \to 0$ as $\nu \to \infty$ . A sequence $\{ \phi_\nu \}_{\nu=1}^{\infty}$ is said to be a Cauchy sequence in $B_{\lambda, a, b}$ if $\beta^\lambda_{a, b, k, k'} (\phi_\nu - \phi_\mu)$ tends to zero as $\nu$ and $\mu$ tend to $\infty$ independently for each pair of $k$ and $k'$. Every Cauchy-sequence in the space $B_{\lambda, a, b}$ is convergent, i.e. $B_{\lambda, a, b}$ is sequentially complete. Moreover, it is locally convex Hausdorff topological vector space. Further, it is a testing function space. The space $D(I)$ is a subspace of $B_{\lambda, a, b}$ and the convergence concept in $D(I)$ is stronger than that in $B_{\lambda, a, b}$. Therefore, the restriction of any $f \in B'_{\lambda, a, b}$ to $D(I)$ is in $D'(I)$, where $B'_{\lambda, a, b}$ is the space of all continuous linear functionals on $B_{\lambda, a, b}$ and called the dual space of $B_{\lambda, a, b}$ .

If $a \leqslant c$ and $0 < d < b$, then $B_{\lambda,c,d}$ is a subspace of $B_{\lambda,a,b}$ and the topology of $B_{\lambda,c,d}$ is stronger than the induced topology on it by $B_{\lambda,a,b}$. Hence, the restriction of $f \in B'_{\lambda,a,b}$ to $B_{\lambda,c,d}$ is in $B'_{\lambda,c,d}$.

Now, we define the countable-union space $B(\lambda, w, z)$ as follows: Let $w$ denote either a finite real number or $-\infty$, and $z$ denote either a finite positive real number or $+\infty$. Choose two monotonic sequences of real numbers $\{a_\nu\}_{\nu=1}^{\infty}$ and $\{b_\nu\}_{\nu=1}^{\infty}$ such that $a_\nu \to w^+$ and $b_\nu > 0, b_\nu \to z^-$. By the above property, $\{B_{\lambda, a_\nu, b_\nu}\}_{\nu=1}^{\infty}$ is a sequence of testing function spaces such that $B_{\lambda, a_\nu, b_\nu} \subset B_{\lambda, a_{\nu+1}, b_{\nu+1}}$ and the topology of $B_{\lambda, a_\nu, b_\nu}$ is stronger than the induced topology on it by $B_{\lambda, a_{\nu+1}, b_{\nu+1}}$ for all $\nu$. Let $B(\lambda, w, z)$ denote the union of these spaces: $B(\lambda, w, z) = \bigcup_{\nu=1}^{\infty} B_{\lambda, a_\nu, b_\nu}$. $B(\lambda, w, z)$ is a vector space. A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to converge in $B(\lambda, w, z)$ to $\phi$ if all the $\phi_\nu$ and $\phi$ belong to $B_{\lambda, a_\nu, b_\nu}$ for some particular $\nu$ and $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to $\phi$ in $B_{\lambda, a_\nu, b_\nu}$ [Zamanian 1968, p.15]. The space $B(\lambda, w, z)$ is called the countable-union space. Since all the $B_{\lambda, a_\nu, b_\nu}$ are sequentially complete, $B(\lambda, w, z)$ is also complete. If $w < u$ and $0 < v < z$, then $B(\lambda, u, v)$ is a subspace of $B(\lambda, w, z)$ and the convergence in $B(\lambda, u, v)$ implies the convergence in $B(\lambda, w, z)$. Therefore, the restriction of any $f \in B'(\lambda, w, z)$ to $B(\lambda, u, v)$ is in $B'(\lambda, u, v)$.

*Lemma* 2.1—Let $\Omega$ be a subset of $C^2$ defined as follows:

$$\Omega = \{(u, v) \mid \operatorname{Re} u > a, \ |\operatorname{Im} v| < b, \ v \neq 0 \text{ or negative integer}\}. \quad .. \quad (2{\cdot}2)$$

For any $(u,v) \in \Omega$, $e^{-ux} \sqrt{vy}\, J_\lambda (vy)$ is a member of $B_{\lambda, a, b}$ whenever $\lambda > -\frac{1}{2}$.

Proof : Let $\lambda > -\frac{1}{2}$ and $(u,v) \in \Omega$. For any pair of non-negative integers $k$ and $k'$,

$$D_x^k \, S_{\lambda,y}^{k'} \, [e^{-ux} \sqrt{vy}\, J_\lambda (vy)] = (-1)^{k+k'} u^k v^{2k} e^{-ux} \sqrt{vy}\, J_\lambda (vy).$$

Hence,

$$\begin{aligned} & e^{ax-by} y^{-\lambda-1/2} D_x^k S_{\lambda,y}^{k'} [e^{-ux} \sqrt{vy}\, J_\lambda (vy)] \\ & = (-1)^{k+k'} e^{ax-by} y^{-\lambda-1/2} u^k v^{2k'} e^{-ux} \sqrt{vy}\, J_\lambda (vy) \\ & = (-1)^{k+k'} u^k v^{2k'} e^{-(u-a)x} e^{-by} y^{-\lambda-1/2} \sqrt{vy}\, J_\lambda (vy). \quad .. \quad (2{\cdot}3) \end{aligned}$$

We see that by the asymptotic expansion of $\mathcal{J}_\lambda(vy)$ and the fact that Re $u > a$, $b > 0$ and $\lambda > -1/2$, the right-hand side of (2·3) is bounded on $I$. Thus, for each pair of non- negative integers $k$ and $k'$,

$$\beta^\lambda_{a,b,k,k'}(-e^{-ux}\sqrt{vy}\,\mathcal{J}_\lambda(vy)) \text{ exists. Therefore, for } (u,v)\in\Omega \text{ and } \lambda > -1/2,$$
$$e^{-ux}\sqrt{vy}\,\mathcal{J}_\lambda(vy)\in B_{\lambda,a,b}.$$

*Remark*: Since $e^{-ux}\sqrt{vy}\,\mathcal{J}_\lambda(vy)\in B_{\lambda,a,b}$ for all $a > w$ and $0 < b < z$, $e^{-ux}\sqrt{vy}\,\mathcal{J}_\lambda(vy)$ belongs to $B\ (\lambda, w, z)$ whenever Re $u > w$, $|\text{Im } v| < z$, $v \neq 0$ or negative integer and $\lambda > -1/2$.

*Lemma* 2.2—If $f(x,y)$ is locally integrable function on $I$ such that $e^{-ax+by}y^{\lambda+1/2}$ $f(x,y)$ is absolutely integrable on $I$, then $f(x,y)$ generates a regular generalized function $f$ in $B'_{\lambda,a,b}$ through the definition

$$<f,\phi> = \int_0^\infty\int_0^\infty f(x,y)\,\phi(x,y)\,dxdy,\ \phi\in B_{\lambda,a,b}, \qquad (2\cdot4)$$

Proof : The proof of the lemma follows from the fact

$$|<f,\phi>| \leqslant \int_0^\infty\int_0^\infty e^{-ax+by}y^{\lambda+\frac{1}{2}} f(x,y)\,dxdy \times \beta^\lambda_{a,b,0,0}(\phi),\quad \phi\in B_{\lambda,a,b}.$$

*Remark*: If the above conditions on $f$ are satisfied for all $a > w$ and $0 < b < z$, then $f$ also generates a regular generalized function in $B(\lambda,w,z)$ through (2·4).

## 3. Generalized Laplace-Hankel Transformation

Let $\lambda$ be restricted as $-\frac{1}{2} < \lambda < \infty$. A generalized function $f$ is said to be complex Laplace-Hankel transformuble if $f\in B'(\lambda,w,z)$ for some real numbers $w$ and $z$, where $z$ is positive. Let $\sigma_f$ be infimum of all such $w$ and $\rho_f$ be supremum of all such $z$. Define $\Omega_f$ in $C^2$ as follows:

$$\Omega_f = \{(u,v)\in C^2 \mid \text{Re } u > \sigma_f,\ |\text{Im } v| < \rho_f,\ v\neq 0 \text{ or a negative integer}\} \qquad (3\cdot1)$$

If $f$ is Laplace-Hankel transformable generalized function, then we see that there exist a pair of real numbers $\sigma_f$ and $\rho_f$, $(\rho_f > 0)$ such that $f\in B'(\lambda,w,z)$ for all $w > \sigma_f$ and $0 < z < \rho_f$ and $f\notin B'(\lambda,w,z)$ if $w < \sigma_f$ and $z > \rho_f$.

We define the complex Laplace-Hankel transform $LH'_\lambda(f)$ of $f$ as the application of $f(x,y)$ to the kernal $e^{-ux}\sqrt{vy}\,\mathcal{J}_\lambda(vy)$ i.e.

$$F(u,v) = (LH'_\lambda f)(u,v) = <f(x,y),\ e^{-ux}\sqrt{vy}\,\mathcal{J}_\lambda(vy)> \qquad (3\cdot2)$$

where $(u, v) \in \Omega_f$.

Since $e^{-ux} \sqrt{vy}\, J_\lambda (vy) \in B_{\lambda, a, b}$ for all $a > \sigma_f$ and $0 < b < \rho_f$, the right-hand side of (3·2) has meaning.

$\Omega_f$ is called the region of definition for $F(u, v)$ and $\sigma_f$ and $\rho_f$ are called the the abscissas of definitions. If the function $f(x, y)$ satisfies the condition of the Lemma 2.2 for all $a > \sigma_f$ and $0 < b < \rho_f$, then the conventional Laplace-Hankel transform of $f$, i.e. $\int_0^\infty \int_0^\infty f(x, y)\, e^{-ux} \sqrt{vy}\, J_\lambda (vy)\, dxdy$ exists for at least one $(u, v) \in \Omega_f$ and $\lambda > -\frac{1}{2}$ and can be identified with the generalized Laplace-Hankel transform (3·2)

Now, we shall prove the analyticity theorem for the generalized Laplace-Hankel transform.

*Theorem* 3.1—If $(LH'_\lambda f)(u, v) = F(u, v)$ for $(u, v) \in \Omega_f$, then $F(u, v)$ is analytic on $\Omega_f$ and

$$DF(u, v) = \langle f(x, y), D\, (e^{-ux} \sqrt{vy}\, J_\lambda (vy)) \rangle \qquad (3\cdot 3)$$

where $D = \dfrac{\partial}{\partial x}$ or $\dfrac{\partial}{\partial y}$.

Proof : For every point $(u', v')$ in $\Omega_f$, each of the functions $F(u, v')$ and $F(u', v)$ is analytic in the single variable $u$ and $v$ respectively in the neighbourhood of $(u', v')$. The proof of this is very similar to that given by Zemanian (1968, p. 58) and Koh and Zemanian (1968), therefore, the details are omitted. Invoking the Hartog's theorem (Bochner and Martin 1948, p. 140) we see that $F(u, v)$ is analytic on $\Omega_f$.

## 4. Inversion Formula

We shall derive the inversion formula for the generalized Laplace-Hankel transformation. For this purpose, first of all we shall state two lemmas without proofs. The proofs are parallel to Lemmas given by Zemanian (1968, p. 64 and p. 180).

*Lemma* 4.1—Let $(LH'_\lambda f)(u, v) = F(u, v)$ for $(u, v) \in \Omega_f$ For $\phi(x, y) \in D(I)$ set

$$\Phi(u, v) = \int_0^\infty \int_0^\infty \phi(x, y)\, e^{ux} \sqrt{vy}\, J_\lambda (vy)\, dxdy \qquad (4\cdot 1)$$

Then, for any pair of fixed numbers $r$ and $r'$ $0 < r < \infty$ , $0 < r' < \infty$ ,

$$\int_{-r}^{r} \int_0^{r'} \Phi(u, v) \langle f(t, T), e^{-ut} \sqrt{vT}\, J_\lambda (vT) \rangle\, dvdw$$

$$= \langle f(t, T), \int_{-r}^{r} \int_0^{r'} \Phi(u, v)\, e^{-ut} \sqrt{vT}\, J_\lambda (vT)\, dudw \rangle . \qquad (4\cdot 2)$$

where $u = \sigma + i\,w$.

*Lemma* 4.2—For $\phi \in D(I)$, set $\Phi\,(u, v)$ as above for Re $u > a$ and $|\,\mathrm{Im} v\,| < b$, $v \neq 0$ or negative integer, then

$$M_{r,\,r'}\ (t,\ T)\ =\frac{1}{2\pi}\int_{-r}^{r}\int_{0}^{r'}\Phi\ (u,\, v)\ e^{-ut}\ \sqrt{vT}\ \mathcal{J}_{\lambda}\ (vT)\ dvdu$$

converges in $B_{\lambda,\,a,\,b}$ to $\phi\ (t,\ T)$ as $r,\ r' \longrightarrow \infty$ .. .. .. (4·3)

*Theorem* 4.1 (Inversion Theorem)—Let $F(u,\ v)\ =\ (LH'_{\lambda}\, f)\ (u,\ v)$ for $(u.\ v) \in \Omega_f$. Then, in the sense of convergence in $D'\ (I)$

$$f\ (x, y)\ =\ \lim_{\substack{r\ \to\ \infty \\ \to\ \infty}}\ \frac{1}{2\pi i}\int_{\sigma - ir}^{\sigma + ir}\int_{0}^{r'} e^{ux}\ \sqrt{vy}\ \mathcal{J}_{\lambda}\ (vy)\ F\ (u,\ v)\ dvdu \qquad (4{\cdot}4)$$

where $\sigma$ is any fixed real number such that $\sigma > \sigma_f$.

PROOF : Let $\phi\ (x, y) \in D(I)$. We shall show that

$$< \frac{1}{2\pi}\int_{\sigma - ir}^{\sigma + ir}\int_{0}^{r'} e^{ux}\ \sqrt{vy}\ \mathcal{J}_{\lambda}\ (vy)\ F(u,\ v)\ dvdw,\ \phi\ (x, y) > \qquad ..\qquad (4{\cdot}5)$$

tends to $< f(x, y),\ \phi\ (x, y) >$ as $r,\ r' \to \infty$ .

From the analyticity of $F\,(u,\ v)$ on $\Omega_f$ the integral in (4·5) is a continuous function of $(x,\ y)$. Therefore, we write (4·5) as

$$\frac{1}{2\pi}\int_{0}^{\infty}\int_{0}^{\infty}\phi\ (x, y)\int_{-r}^{r}\int_{0}^{r'} e^{ux}\ \sqrt{vy}\ \mathcal{J}_{\lambda}\ (vy)\ F\ (u,\ v)\ dvdw\ dydx.$$

Since $\phi\ (x, y)$ is of compact support and the integrand is a continuous function of $(x, y;\ w,\ v)$, the order of integration may be changed. This yields

$$\frac{1}{2\pi}\int_{-r}^{r}\int_{0}^{r'}\ < f\ (t,\ T),\ e^{-ut}\ \sqrt{vT}\ \mathcal{J}_{\lambda}\ (vT) > \ \Phi\ (u,\ v)\ dvdw$$

where $\Phi\,(u, v)$ is as in Lemma 4.1. Hence, by Lemma 4.1, this is equal to

$$< f\,(t, T),\ \frac{1}{2\pi} \int_{-r} \int_{0}^{r'} e^{-ut}\ \sqrt{vT}\ \mathcal{J}_\lambda\,(vT)\ \Phi\,(u, v)\ dvdw > . \qquad .. \qquad (4{\cdot}6)$$

Again, by Lemma 4.2, eqn. (4·6) converges to $< f\,(t, T).\ \phi\ (t, T) >$ as $r$ and $r'$ tend to $\infty$ . This completes the proof of the theorem.

*Theorem* 4·2—(Uniqueness Theorem) : If $(LH'_\lambda f)\ (u, v) = F\,(u, v)$ for $(u, v) \in \Omega_f$ and $(LH'_\lambda\, g)\ (u, v) = G(u, v)$ for $(u, v) \in \Omega_g$ if $\Omega_f \cap \Omega_g$ is not empty and if $F\,(u, v) = G(u, v)$ on $\Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality in $D'(I)$.

PROOF : By inversion theorem,

$$f - g = \lim_{\substack{r \to \infty \\ r' \to \infty}} \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} \int_{0}^{r'} e^{ux}\ \sqrt{vy}\ \mathcal{J}_\lambda\,(vy)$$

$$\times\ [F\,(u, v) - G(u, v)]\ dvdu$$

$$= 0$$

Thus, $f = g$ in $D'(I)$.

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## REFERENCES

Bhonsle, B.R., and Choudhary, M.S. (in press). The generalized Laplace-Hankel transformation. Communicated to *Bull. Calcutta math. Soc.*

Bochner, S., and Martin, W.T. (1948). Several Complex Variables. Princeton University Press.

Zemanian, A.H. (1968). The Generalized Integral Transformations. John Willey and Sons, New York.

Zemanian, A.H., and Koh, E.L. (1968). The complex Hankel and *I*-transformations of generalized functions. *S I A M. J. appl. Math.*, **16,** 945.