A NUMERICAL METHOD FOR SOLVING MILDLY NONLINEAR ELLIPTIC PROBLEMS OVER IRREGULAR REGIONS

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This paper presents an approach of quasilinearization and invariant imbedding techniques for finding the numerical solution of mildly nonlinear elliptic partial differential equations over arbitrary irregular regions. The given irregular region is imbedded in a conveniently chosen rectangular region and the original nonlinear problem is replaced by a sequence of linear problems obtained by employing quasilinearization. This sequence of linear problems subject to appropriate boundary conditions on the rectangle and some internal constraints is then solved using invariant imbedding procedures and the method of Lagrange multipliers. A numerical example is solved to illustrate the method.

1. Introduction

The quasilinearization method, which was first developed by Bellman (1955) and Kalaba (1959) has been shown to be a versatile tool for solving nonlinear ordinary and partial differential equations by Bellman and Kalaba (1965). The method consists in solving, instead of the original nonlinear problem, a sequence of linear problems which converges quadratically to the solution of the nonlinear problem assuming convergence. On the other hand, the use of invariant imbedding approach (Bellman *et al.* 1967 and Angel and Bellman 1972) allows us to replace linear boundary value problems, which often have unfavourable stability properties, by stable initial value problems which can be solved by well known numerical techniques.

Recently some authors in a series of papers (Angel 1973, Buzbee *et al.* 1971, George 1970) have studied the solution of linear elliptic equations over irregular regions by imbedding the given region into a rectangular region. This idea of imbedding was first used by George (1970) in conjunction with a direct method of Buzbee *et al.* (1970) for the Poisson equation. Unfortunately, this method is found to have a limited applicability to linear equations with constant coefficients. Dynamic programming approach has been used by Angel (1973) to obtain the numerical solution of a general class of linear elliptic equations over irregular regions.

In this paper, we present a combined approach of quasilinearization and invariant imbedding techniques to solve mildly nonlinear elliptic partial differential equations over irregular regions. The given region is imbedded in a conveniently chosen rectangular region. The original nonlinear problem is replaced by a sequence of linear problems obtained by employing quasilinearization. This sequence of linear problems is then solved successively subject to appropriate boundary conditions on the imbedding rectangle and the internal constraints by invariant imbedding approach and the method of Lagrange multipliers. The method is repeated till the sequence of numerical solutions converges. An example is solved to illustrate the method.

2. Quasilinearization

Consider the following nonlinear elliptic problem:

$$u_{xx} + u_{yy} = f(u) \quad \text{on } S \quad \ldots \quad \ldots \quad (1)$$

$$u = h(x, y) \quad \text{on } \partial S \quad \ldots \quad \ldots \quad (2)$$

where S is an arbitrary, simply connected closed region, ∂S is its boundary, and $h(x, y)$ is a prescribed function on the boundary.

Applying quasilinearization to (1), we get the following sequence:

$$u_{xx}^{(n+1)} + u_{yy}^{(n+1)} = f[u^{(n)}] + [u^{(n+1)} - u^{(n)}] f_u [u^{(n)}],$$
$$(n=0,1,2,..) \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$u^{(n)}$ denotes the value of u at the nth iteration, $u^{(0)}$ is an initial guess for u.

The boundary condition (2) can be rewritten as

$$u^{(n+1)} = h(x, y) \quad \text{on } \partial S. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Bellman and Kalaba (1965) have shown that, for well behaved functions $f(u)$, the solutions obtained from the sequence (3) and (4) will converge for large n and, hence, converge to the solution of the original nonlinear problem (1) and (2) provided the initial guess, $u^{(0)}$, is sufficiently close to the solution. Considering the solution of sequence (3) and (4) at the nth step, we have

$$v_{xx} + v_{yy} = f(w) + (v - w) f_4 (w) \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

subject to

$$v = h(x, y) \text{ on } \partial S \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where w is the known solution at the $(n-1)$th step and v is the solution to be found at the nth step.

Let the given region S be imbedded into a conveniently chosen rectangular region R with its boundary as ∂R so that $S \subset R$. Suppose v is arbitrarily specified at those boundary points of ∂R which are given by

$$v = d(x,y) \text{ on } \partial R - (\partial R \cap \partial S) \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where $d(x,y)$ is to be chosen appropriately. Hence our problem reduces to finding the solution of

$$v_{xx} + v_{yy} = f(w) + (v - w) f_u (w) \quad \ldots \quad \ldots \quad (8)$$

subject to

$$v = g(x, y) \text{ on } \partial R \quad .. \quad .. \quad .. \quad (9)$$

where
$$g(x,y) = \begin{cases} h(x,y) & \text{on } \partial S \cap \partial R \\ d(x, y) & \text{on } \partial R—(\partial R \cap \partial S). \end{cases} \quad .. \quad .. \quad (10)$$

Here 1 is chosen arbitrarily with the restriction that g be smooth on ∂R. Given eqns. (8) and (9), the aim is to find the solution $v(x, y)$ in such a way that

$$v(x', y') = h\,(x', y') \quad .. \quad .. \quad .. \quad (11)$$

on ∂S, where (x', y') is a point on the boundary ∂S. Hence our new problem is defined by the eqns. (8), (9), and (11). If this problem can be solved successively, its solution will converge to the solution of the given problem (1) and (2) for large n, provided $u^{(0)}$ is chosen appropriately.

3. Discretization

Let us take the rectangular region R as $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and discretize the region R by taking $a = ph$; $b = mh$ where p, m are positive integers. We use the notation $v_{i,j} = v\,(ih, jh)$ and define

$$v_i = \begin{bmatrix} v_{i,1} \\ v_{i,2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_{i,m-1} \end{bmatrix} \quad .. \quad .. \quad .. \quad (12)$$

for $i = 1, 2, \ldots\ldots, p—1$.

Now replacing the partial derivatives in (8) by standard five point finite difference approximations, we obtain,

$$— v_{i+1} + 2\, v_i — v_{i-1} + Q\, v_i — r_i = —f_i —fd_i\, (v_i — w_i)$$
$$(\, i = 1, 2, \ldots\ldots, p-1), \quad .. \quad .. \quad .. \quad (13)$$

where the matrix Q is defined as

$$Q = (q_{ij}) \qquad q_{ij} = \begin{cases} 2 \;; & i = j \\ -1 \;; & |\, i—j \,| = 1 \\ 0 \;; & \text{otherwise,} \end{cases} \quad .. \quad (14)$$

the vector r_i is defined as

$$r_i = (r_{ij}); \qquad r_{ij} = \begin{cases} v_{i0} & ; \quad j = 1 \\ v_{i,m} & ; \quad j = m - 1 \\ 0 & ; \quad \text{otherwise,} \end{cases} \qquad \cdots \quad (15)$$

the vector f_i is defined as

$$f_i = \begin{bmatrix} h^2 f_{i,1} \\ h^2 f_{i,2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ h^2 f_{i,m-1} \end{bmatrix} \; ; \; i = 1,2,..,p-1 \qquad \cdots \qquad \cdots \quad (16)$$

the diagonal matrix fd_i is defined as

$$fd_i = \begin{bmatrix} h^2 fd_{i,1} & & & & 0 \\ & h^2 fd_{i,2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & h^2 fd_{i,m-1} \end{bmatrix} \qquad \cdots \quad (17)$$

(i=1,2,......, p—1), where $f_{i,j's}$ and $fd_{i,j's}$ (j=1,2,......, m—1) are obtained by evaluating the functions f and fd at the known solution w, respectively.

The equation (13) is subject to two point boundary conditions v_o and v_p which are known from the boundary conditions. Furthermore, the eqn. (11) can be written as

$$v_{i,k} = h_{i,k} \; ; \quad k \in k_j \qquad \cdots \qquad \cdots \qquad \cdots \quad (18)$$

where k_j represents the set of the y-ordinates where the grid points at $x = ih$ intersect the curve ∂S — ($\partial S \cap \partial R$). The eqn. (18) can also be written in matrix-vector form as

$$T_i v_i = h_i \quad ; \quad i \in I \subset (1, 2 \ldots, p-1), \qquad \cdots \qquad \cdots \qquad \cdots \quad (19)$$

where T_i are appropriate rectangular matrices containing only zeroes and ones. Martices T_i are of order $s_i x$ (m—1) where s_i denotes the number of the grid points at $x = ih$ intersecting with the curve ∂S—($\partial S \cap \partial R$) and rank of T_i is s_i. Hence the discrete problem is to find the solution of the matrix-vector equation (13) subject to the boundary conditions v_o andt v_n and the constraints given by eqn. (19).

4. Invariant Imbedding

We seek a solution to (13) in the form

$$v_{i+1} = A_i v_i + b_i \qquad (20)$$

where the matrix A_i and the vector b_i are independent of v_i. The existence of solutions of this form is a consequence of the linearity of (13). Further, the justification for the step is given by Angel and Bellman (1972). In order to find the recurrence relations for matrices A_i and vectors b_i, let us assume that $i \notin I$ initially. In other words, when eqn. (13) is not constrained by eqn. (19), then, on eiminating v_{i+1} between (20) and (13) and solving for v_i, we obtain

$$v_i = (2I + Q - A_i + fd_i)^{-1} (v_{i-1} + b_i + r_i - f_i + fd_i w_i) \qquad (21)$$

By comparing (21) with (20),

and
$$\left.\begin{aligned} A_{i-1} &= (2I + Q - A_i + fd_i)^{-1} \\ b_{i-1} &= A_{i-1} (b_i + r_i - f_i + fd_i w_i). \end{aligned}\right\} \qquad (22)$$

In case $i \in I$, that is, when constraints of eqn. (19) appear in eqn. (13)one can proceed in such a way that the solution sought in the form of (20) satisfies both the eqns. (13) and (19). This can be accomplished by the use of Lagrange multipliers. We write eqn. (13) as

$$-v_{i+1} + 2v_i - v_{i-1} + Qv_i - r_i = -f_i - fd_i (v_i - w_i) - T_i' \lambda_i \qquad (23)$$

where the prime denotes the matrix transpose and λ_i's are the Lagrange multipliers corresponding to the internal grid points interesecting with the curve ∂S—$(\partial R \cap \partial S)$. To determine these Lagrange multipliers in such a way that the solution to eqn. (13) sought in the form of (20), satisfies the constraints of eqn. (19), we eliminate v_{i+1} between (20) and (23) and solve for v_i:

$$v_i = (2I + Q - A_i + fd_i)^{-1} (v_{i-1} + b_i + r_i - f_i + fd_i w_i - T_i' \lambda_i) \qquad (24)$$

Now substituting this value of v_i in (19),

$$T_i F_i^{-1} (v_{i-1} + b_i + r_i - f_i + fd_i w_i - T_i' \lambda_i) = h_i \qquad (25)$$

where

$$F_i = (2I + Q - A_i + fd_i) \qquad (26)$$

Solving eqn. (25) for λ_i,

$$\lambda_i = (T_i F_i^{-1} T_i')^{-1} T_i F_i^{-1} (v_{i-1} + b_i + r_i - f_i + fd_i w_i) - (T_i F_i^{-1} T_i')^{-1} h_i \quad (27)$$

or

$$\lambda_i = B_i (v_{i-1} + b_i + r_i - f_i + fd_i w_i) - C_i h_i \quad (28)$$

where

$$\left.\begin{aligned} B_i &= (T_i F_i^{-1} T_i')^{-1} T_i F_i^{-1} \\ \text{and} \quad C_i &= (T_i F_i^{-1} T_i')^{-1}. \end{aligned}\right\} \quad (29)$$

Now substituting the value of λ_i given by eqn. (28) in eqn. (24), we find

$$v_i = F_i^{-1} \{I - T_i' B_i\} v_{i-1} + F_i^{-1} \{(I - T_i' B_i)(b_i + r_i - f_i + fd_i w_i) + T_i' C_i h_i\} \quad (30)$$

Comparing it with (20), one gets,

$$\left.\begin{aligned} A_{i-1} &= F_i^{-1} \{I - T_i' B_i\} \\ \text{and} \quad b_{i-1} &= A_{i-1} (b_i + r_i - f_i + fd_i w_i) + F_i^{-1} T_i' C_i h_i \end{aligned}\right\} \quad (31)$$

with the conditions

$$A_p = 0 \quad \text{and} \quad b_p = v_p \quad (0 \text{ is the null matrix}) \quad (32)$$

which are obtained by considering (20) for $i = p-1$.

Thus the calculation procedure is to compute the matrices A_i and the vectors b_i, ($i = p-1, \ldots\ldots, 1, 0$) recursively starting with eqn. (32) and employing either eqn. (22) (when the constraints of eqn. (19) do not arise) or eqn. (31) (when the constraints arise), and to store the intermediate values of A_i and b_i. Once this has been done, the vectors v_i are determined from eqn. (20) in a forward sweep, starting with v_0 known from the boundary condition and using the stored values of A_i and b_i. The vectors v_i just computed serve the approximation to the solution for the next iteration. The above procedure is repeated till the convergence is reached. The execution of the algorithm is stopped when the difference of the solutions between two consecutive iterations is within a prescribed tolerance limit.

Or,
$$\max_{i,j} |v_{ij} - w_{ij}| \leqslant 10^{-k}; \quad \begin{aligned} i &= 1,2,\ldots\ldots, p-1 \\ j &= 1,2,\ldots\ldots, m-1 \end{aligned} \quad (33)$$

where k is a positive integer.

5. Nonsingularity and Stability

We now consider the existence of inverses of the matrices appearing in the recurrence relations (22) and (31).

Case I : Suppose $i \notin I$

It can easily be shown that the matrix Q is positive definite. Since Q is a real symmetric matrix, we can write

$$Q = P\,M\,P'$$

where P is an orthogonal matrix and M is the diagonal matrix

$$M = \begin{bmatrix} \mu_1 & & & & O \\ & \mu_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ O & & & & \mu_{m-1} \end{bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

where $\mu_j > 0$, $(j=1,2,\ldots\ldots, m-1)$ .. .. .. (36)

We assume that the matrix $2I + Q + fd_i$ is positive definite. Inductively, we find that A_i can be written as

$$A_i = P\,Q_i\,P' \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where P is the orthogonal matrix of (34) and Q_i is the diagonal matrix.

$$Q = \begin{bmatrix} \xi_1^{(i)} & & & & O \\ & \xi_2^{(i)} & & & \\ & & \ddots & & \\ & & & \ddots & \\ O & & & & \xi_{m-1}^{(i)} \end{bmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Then by (22) and (31), the characteristic values of A_i must satisfy

$$\xi_j^{(i-1)} = \frac{1}{2 + \mu_j - \xi_j^{(i)} + \zeta_j^{(i)}}\,;\ j=1,2,\ldots\ldots,m-1,\ (i \notin I) \quad \ldots \quad (39)$$

with $\xi_{j=0,}^{(p-1)}$ $(j = 1, 2, \ldots\ldots, m-1)$ .. .. .. (40)

where $\zeta_j^{(i)}$, $(j=1,2,\ldots\ldots, m-1)$ are the characteristic values of fd_i .

Again proceeding inductively, we find,

$0 < \xi_j^{(i)} < 1, \quad j = (1,2,\ldots\ldots, m-1)\ (i \notin I),$ .. .. (41)

Therefore, the matrices F_i, are positive definite, and hence the inverses exist.

Case II : $i \in I$

To show that the inverses of the matrices occurring in the recurrence relations (32) exist, the explanation given above holds good for the materices F_i. In order to prove that the matrices

$$C_i = (T_i\, F_i^{-1}\, T_i')\text{ are invertible,}$$

we require the following lemma :

Let $M \in R^{nxn}$ be a symmetric positive definite matrix.

Let $P \in R^{kxn}$, $k \leqslant n$, be of full rank and suppose $p_{ij} \in \{0, 1\}$. If the rows of P are orthogonal, then PMP′ is positive definite.

For the proof of this result, the reader is referred to Angel (1973). Clearly the matrices, T_i, enjoy the same properties as that of P in the above lemma and furthermore, as previously noted, the matrices F_i are positive definite, Moreover, simple observation will reveal that $F_i's$ are symmetric. Hence applying the above lemma, we find that the matrices $T_i\, F^{-1}\, T_i'$ are positive definite and, hence, nonsingular. In addition to this, using a similar type of method as used in Angel (1973), it can be shown that the matrices A_i have a spectral radius bounded by unity. Thus, the stability of the computational scheme can be established.

6. Numerical Example

To illustrate the method, we consider an example :

$u_{xx} + u_{yy} = e^u$ .. .. .. (42)

is an equation of interest in magneto-hydrodynamics and radiation.We solve it in two circular domains of radii 1/2 and 1/4 with $u = 1$ on the boundary of the circles. The circles are imbedded in a unit square in the first case and a square of length 1/2 in the 2nd case. The squared domains are discretized taking two different step sizes in each case. The values of u on all possible points on the boundary of the imbedding domains, namely the squares, are also taken to be equal to 1 as consistent with the original boundary condition. To start the process, the initial guess to the solution is taken as $u = 1$ at all internal mesh points. Only a few iterations were required for the method to converge upto six significant places. The results of the above example for two circular domains of radii 1/2 and 1/4 are given in Tables I and II respectively.

TABLE I

Comparison of computed values at certain mesh points with two mesh sizes ($h=1/10$, $h=1/20$)

Mesh Points	Computed values of u when $h=1/10$	Computed values of u when $h = 1/20$
(1/10, 1/10)	0·988314	0·992535
(1/10, 6/10)	0·961632	0·941726
(2/10, 1/10)	1·000000	1·000000
(2/10, 8/10)	0·958837	0·951972
(3/10, 1/10)	1·000000	1·000000
(3/10, 2/10)	0·940655	0·917940
(3/10, 6/10)	0·890323	0·870425
(4/10, 5/10)	0·866296	0·847063
(5/10, 3/10)	0·883923	0·864604
(5/10, 9/10)	0·952574	0·937919
(6/10, 2/10)	0·920117	0·899876
(6/10, 4/10)	0·872328	0·852876
(7/10, 3/10)	0·909282	0·888218
(8/10, 1/10)	1·000000	1·000000
(8/10, 2/10)	0·958837	0·951972
(9/10, 2/10)	0·980126	0·971181

TABLE II

Comparison of computed values at certain mesh points with two mesh sizes ($h = 1/20$, $h = 1/40$)

Mesh Points	Computed values of u when $h = 1/20$	Computed values of u when $h = 1/40$
(1/20, 1/20)	0·996707	0·997390
(1/20, 2/20)	0·996119	0·997601
(2/20, 1/20)	0·997482	0·998612
(2/20, 7/20)	0·988803	0·984755
(3/20, 1/20)	1·000000	1·000000
(3/20, 4/20)	0·977101	0·970660
(4/20, 2/20)	0·986185	0·978707
(4/20, 7/20)	0·977264	0·970697
(5/20, 5/20)	0·968924	0·962413
(6/20, 2/20)	0·986185	0·978707
(6/20, 3/20)	0·977264	0·970697
(7/20, 1/20)	1·000000	1·000000
(7/20, 4/20)	0·977101	0·970660
(7/20, 9/20)	1·000000	1·000000
(8/20, 1/20)	0·997482	0·998612
(8/20, 3/20)	0·988803	0·984755
(8/20, 5/20)	0·983586	0·976518
(9/20, 1/20)	0·996707	0·997390
(9/20, 2/20)	0·996119	0·997601

7. Discussion

The combined use of quasilinearization and invariant imbedding seems to be well suited for solving mildly nonlinear elliptic partial differential equations over arbitrary irregular domains. The quasilinearization procedure replaces the original nonlinear equation by a sequence of linear equations and invariant imbedding procedure is exploited to solve these linear boundary value problems as initial value problems. The advantage in using quasilinearization procedure lies in the fact that it has quadratic convergence provided that the initial guess is sufficiently close to the exact solution. However, one can usually obtain a reasonable crude initial guess for most of the physical problems. The advantage in the use of invariant imbedding approach is to avoid two point boundary value problems, which often have associated unstable computational algorithms, and replace them by appropriate initial value problems which possess simple stable algorithms. Of more importance is the fact that irregular regions can easily be handled by such a method. This method can also be extended to the solution of nonlinear elliptic partial differential equations. This type of work is under investigation.

In the example considered in this paper, the discretization of the differential equation has been done by using central differences (five point difference analogue). The truncation error for such a scheme is of the order of h^2. On examining the results for different mesh sizes included in Tables I and II, one finds that the computed values improve as the mesh is made finer.

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