ON MODIFIED DEFICIENT VALUES OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVES

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A number of results concerning the value distribution of meromorphic functions of finite or infinite order, in terms of modified characteristic function and modified deficient values have been obtained. As a consequence of the main theorems, numerous results in the form of corollaries have been deduced, some of which extend the earlier results obtained by Pfluger, Shah and Singh, Ozawa, Toda, etc.

1. INTRODUCTION

For meromorphic functions, in the finite plane of infinite order the so called 'exceptional set' appears in the second fundamental theorem of Nevanlinna. Therefore, in contrast to the functions of finite order, we have many difficulties and troubles in the investigation of value distribution of meromorphic functions of inifinite order. To avoid some of them, very recently, Toda (1970) has introduced some new concepts like modified characteristic function, deficiencies, etc., in the Nevanlinna theory. Applying these concepts, to the classical cases, he obtained therein some results which hold even for functions of infinite order, though already ascertained for functions of finite order only.

In the present paper, we have obtained a number of results concerning the value distribution of meromorphic functions of finite or infinite order in terms of modified characteristic function etc., some of them improve considerably the results of Toda (1970). We explain our notations in § 2 before stating our results in § 3. In § 4, all those allied results which will be needed in the proof of the results stated in § 3 have been collected in the form of lemmas; the remaining sections have been devoted to the proofs of our results.

2. TERMINOLOGY AND NOTATIONS

We denote by C, the set of all (finite) complex numbers, and by $\bar{C}$, the extended complex plane. Thus $\bar{C}=C \cup \{\infty\}$. We shall consider functions $f(z)$ which are nonconstant meromorphic in C. Throughout the paper we shall assume the familiarity with the standard notations, namely

$$T(r,f),\ m(r,a),\ N(r,a),\ N(r,a),\ S(f,f),$$
$$\delta\ (a,f),\ \triangle\ (a,f),\ \Theta\ (a,f),\ \text{etc. } (a\in\bar{C}), \qquad \ldots \quad \ldots \quad \ldots \qquad (2\cdot1)$$

of the Nevanlinna theory (See Hayman 1964).

The order ρ $(0 \leqslant \rho \leqslant \infty)$ of $f(z)$ is given by

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log \gamma} = \rho.$$

Denote by α any non-negative number smaller than ρ if $\rho > 0$, and zero if $\rho = 0$. For any positive number r_0, define

$$\Psi\ (r,r_0) = \int_{r_0}^{r} \frac{\psi\ (t)}{t^{1+\alpha}}\ dt,$$

where $\psi\ (t)$ is any auxiliary function such as $T(r,f)$,. . . .mentioned in (2·1).

As remarked by Toda (1970) many of the properties of $T_\alpha\ (r,r_0,f)$, $N_\alpha\ (r,r_0,f)$ etc. being independent of r_0, we may omit r_0 in the sequel except when it is necessary, and write only $T_\alpha\ (r,f)$ for $T_\alpha\ (r,r_0,f)$ etc. etc.

The function $T_\alpha\ (r,f)$ is called the modified α-characteristic function of $f(z)$, and $T_\alpha\ (r,f)\ T \to \infty$ as $r \to \infty$.

Further, for any $a \in \bar{C}$, define

$$\delta_\alpha\ (a,f) = \liminf_{r\to\infty} \frac{m_\alpha\ (r,a)}{T_\alpha\ (r,f)}$$

$$\triangle_\alpha\ (a,f) = \limsup_{r\to\infty} \frac{m_\alpha\ (r,a)}{T_\alpha\ (r,f)}\ .$$

In view of the modified first fundamental theorem (Toda 1970, p. 638):

$$T_\alpha\ (r,f) = m_\alpha\ (r,f) + N_\alpha\ (r,f)$$
$$= m_\alpha\ (r,a) + N_\alpha\ (r,a) + \epsilon(r),$$

with

$$\epsilon\ (r) = \begin{cases} O\ (\log r), \text{ for } \alpha = 0 \\ O\ (1), \text{ for } \alpha \neq 0 \end{cases}$$

it is noted that

$$\delta_\alpha (a, f) = 1 - \limsup_{r \to \infty} \frac{N_\alpha (r,a)}{T_\alpha (r,f)}$$

$$\triangle_\alpha (a, f) = 1 - \liminf_{r \to \infty} \frac{N_\alpha (r,a)}{T_\alpha (r, f)} .$$

A value $a \in \overline{C}$ is said to be Nevanlinne α-deficient if $\delta_\alpha (a,f) > 0$, and Valiron α-deficient if $\triangle_\alpha (a,f) > 0$. The quantities $\delta_\alpha (a,f)$ and $\triangle_\alpha(a,f)$ are, respectively, called the modified α-deficiency in sense of Nevanlinna and Valiron. By modified α-deficiency only we shall mean in sense of Nevanlinna (1970) unless specified otherwise.

Also, define

$$\Theta_\alpha (a, f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}_\alpha (r,a)}{T_\alpha (r,f)} .$$

Finally, by the term 'admissible α' we shall mean $\alpha > 0$ and $0 < \rho \leqslant \infty$ or $\alpha = 0$ and $0 \leqslant \rho <$.

3. Statement and Discussion of the Results

Theorem 1—Let $f(z)$ be a transcendental meromorphic function in C. Then, for each integer $l \geqslant 1$,

$$\sum_{a \in C} \delta_\alpha (a, f) \leqslant \lim_{r \to \infty} {\sup \atop \inf} \frac{T_\alpha (r, f^{(l)})}{T_\alpha (r, f)} \leqslant 1 + l \, (1 - \Theta_\alpha (\infty, f)) \qquad .. \qquad (3{\cdot}1)$$

for any admissible α.

Corollary 1·1—If $\sum_{a \in C} \delta_\alpha (a, f) = 1$ and $\Theta_\alpha (\infty, f) = 1$, then

$$T_\alpha (r, f^{(l)}) \sim T_\alpha (r, f), r \to \infty.$$

Corollary 1.2— $$\sum_{a \in C} \delta_\alpha (a, f) \leqslant \delta_\alpha (0, f^{(l)}) [1 + l \, (1 - \Theta_\alpha (\infty, f))]. \qquad (3{\cdot}2)$$

Remark : For $l=1$, the results (3·1) and (3·2) reduce to those of Theorems 1 and 2 of Toda (1970).

Further, Corollary 1·2 immediately confirms the following:

Corollary 1·3—If $\sum_{a \in C} \delta_\alpha (a, f) = 1$ and $\Theta_\alpha (\infty, f) = 1$, then $\delta_\alpha (0, f^{(l)}) = 1$ i.e. $f(z)$ has 0 as a full* α-deficient value for each $l \geqslant 1$.

* By full α-deficient value we mean a value for which $\delta_\alpha (a, f) = 1$.

Also, from (3·1) (for $l=1$), we readily note that if $\sum_{a \in C} \delta_\alpha(a,f) = 2$, then

$$T_\alpha(r,f') \sim 2\, T_\alpha(r,f),\ r \to \infty . \qquad (3\cdot3)$$

It is now natural to think if the result (3·3) can be generalised for higher derivatives. However, we prove the following:

Theorem 2—Let $f(z)$ be meromorphic in C, Let $a,\ b \in C$ be such that $a \neq b$ and $\delta_\alpha(a,f) = 1$ and $\delta_\alpha(b,f) = 1$.

Then

$$T_\alpha(r,f^{(l)}) \sim (l+1)\,T_\alpha(r,f),\ r \to \infty$$

for any admissible α.

Remark : It would be interesting if the above conditions can be replaced by $\sum_{a \in C} \delta_\alpha(a,f) = 2$.

In what follows, we prove a result which is better than the left hand inequality in Theorem 1 above, but for a particular case when $l=1$.

Theorem 3—Let $f(z)$ be a meromorphic function in C. Then

$$\liminf_{r\to\infty} \frac{T_\alpha(r,f')}{T_\alpha(r,f)} \geqslant \sum_{a \in C} \Theta_\alpha(a,f) \qquad (3\cdot4)$$

for any admissible α.

Remark : It would be interesting to extend this result to the higher derivatives.

Corollary 3·1—If $\sum_{a \in C} \Theta_\alpha(a,f) = 2$, then

$$\lim_{r\to\infty} \frac{T_\alpha(r,f')}{T_\alpha(r,f)} = 2 - \Theta_\alpha(\infty, f) \qquad (3\cdot5)$$

$$\lim_{r\to\infty} \frac{\bar{N}_\alpha(r,a)}{T_\alpha(r,f)} = 1 - \Theta_\alpha(a,f),\ \forall\, a \in C \qquad (3\cdot6)$$

$$\Delta_\alpha(a,f) \leqslant \Theta_\alpha(a,f),\quad \forall\, a \in C \qquad (3\cdot7)$$

$$\Delta_\alpha(a,f) = \Theta_\alpha(a,f) \Longleftrightarrow \Delta_\alpha(a,f) = 1 \qquad (3\cdot8)$$

provided $a \in C$ is such that f—a has no simple zeros;

$$\Theta_\alpha(\infty, f') = \frac{1}{2 - \Theta_\alpha(\infty, f)} \qquad (3\cdot9)$$

$$\delta_\alpha(\infty, f') = \frac{\delta_\alpha(\infty, f)}{2 - \Theta_\alpha(\infty, f)} \qquad (3\cdot10)$$

$$\Delta_a(\infty, f') = \frac{\Delta_a(\infty, f)}{2 - \Theta_a(\infty, f)} \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}11)$$

Remark : One may immediately note, from (3·10) and (3·11), that ∞ is a Nevanlinna a -deficient (or Valiron a -deficient) value of f and f' simultaneously.

Corollary 3·2 If $\sum_{a \in C} \delta_a(a, f) = 2$ then

$$\lim_{r \to \infty} \frac{T_a(r, f')}{T_a(r, f)} = 2 - \delta_a(\ \ , f) \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}12)$$

$$\lim_{r \to \infty} \frac{N_a(r, a)}{T_a(r, f)} = \lim_{r \to \infty} \frac{\bar{N}_a(r, a)}{T_a(r, f)} = 1 - \delta_a(a, f), \ \forall\, a \in \bar{C} \quad \ldots \quad (3{\cdot}13)$$

$$\Delta_a(\infty, f') = \delta_a(\infty, f') = \frac{\delta_a(\infty, f)}{2 - \delta_a(\infty, f)} \quad \ldots \quad \ldots \quad (3{\cdot}14)$$

Corollary 3.3 — If $\sum_{a \in C} \delta_a(a, f) = 1$ and $\Theta_a(\infty, f) = 1$, then for each integer $l \geqslant 1$,

$$\delta_a(0, f^{(l)}) = 1 = \Theta_a(\infty, f^{(l)}) \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}15)$$

$$\delta_a(\infty, f^{(l)}) = \delta_a(\infty, f) \text{ and } \Delta_a(\infty, f^{(l)}) = \Delta_a(\infty, f). \quad \ldots \quad (3{\cdot}16)$$

To state the next theorem, more precisely, let us define for any admissible a, the quantity $k_a(f)$ as:

$$k_a(f) = \limsup_{r \to \infty} \frac{N_a(r, \frac{1}{f}) + N_a(r, f)}{T_a(r, f)}$$

Toda has proved that for functions of non-integral order ρ $(0 < \rho < \infty)$, for any admissible a, $k_a(f) \geqslant A(\rho)$, where $A(\rho)$ is a quantity depending on ρ and is positive, (see Toda 1970, Theorem 3). Since the order of $f^{(l)}(z)$ is not changed, we conclude, for a function of non-integral order, that

$$k_a(f^{(l)}) \geqslant A(\rho) > 0,$$

where l is any integer $\geqslant 1$.

Toda has also deduced, "If, for any admissible a, $k_a(f) = 0$, then $f(z)$ is of regular growth and if $\rho < \infty$, then ρ is a positive integer" (see Toda 1970, Corollary 2, p. 654). Consequently, we have

Proposition 1 — If, for any admissible a, $k_a\ (f^{(l)}) = 0$, then $f(z)$ is of regular growth, and if $\rho < \infty$, then ρ is a positive integer.

Further, we prove the following:

Theorem 4—Let $f(z)$ be a transcendental meromorphic function in C. Let $\sum_{a \in C} \delta_a\ (a, f) \geqslant 1 - \gamma$, $\delta_a\ (\infty, f) \geqslant 1 - \gamma \quad (0 < \gamma < 1)$.

Then

$$\limsup_{r \to \infty} \frac{N_a\left(r, \frac{1}{f^{(l)}}\right)}{T_a\ (r, f^{(l)})} \leqslant \frac{(l+1)\ \gamma}{1-\gamma} \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}17)$$

and

$$\limsup_{r \to \infty} \frac{N_a\ (r, f^{(l)})}{T_a\ (r, f^{(l)})} \leqslant \frac{(l+1)\ \gamma}{1+l\ \gamma} \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}18)$$

A fortiori

$$k_a\ (f^{(l)}) \leqslant \frac{(l+1)\ [2+(l-1)\gamma]\gamma}{(1-\gamma)\ (\epsilon+\gamma l)}. \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}19)$$

Corollary 4.1—For any admissible a,

$$k_a\ (f^{(l)}) = 0 \Longleftrightarrow \sum_{a \in C} \delta_a\ (a, f) = 1 \text{ and } \delta_a\ (\infty, f) = 1.$$

Following the technique of Shah and Singh (1959), the result of Corollary 4·1 may be restated as, "For any admissible a,

$$k_a\ (F^{(l)}) = 0 \Longleftrightarrow \sum_{\substack{a \in C \\ a \neq b}} \delta_a\ (a, f) = 1,\ \delta_a\ (b, f) = 1,$$

where $F(z) = (f(z)-b)^{-1}$, $b \in C$." This in view of Proposition 1 above, implies the following

Corollary 4.2—If for any admissible a

$$\sum_{\substack{a \in C \\ a \neq b}} \delta_a\ (a, f) = 1 \text{ and } \delta_a\ (b, f) = 1$$

then, $f(z)$ is of regular growth, and if $\rho < \infty$, then ρ is a positive integer.

Theorem 5—Let $f(z)$ be a meromorphic function in C. Let $a \in C$ be such that $f-a$ has no zero of order $< p$, where p is an integer $\geqslant 2$. Then

$$\delta_a\ (a, f) \leqslant 1-p\ (1-\Theta_a)\ ^{(a,f)} \quad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}20)$$

$$\delta_a (a,f) = \Theta_a (a,f) \Longleftrightarrow \delta_a (a,f) = 1 \qquad \qquad (3{\cdot}21)$$

for any admissible α.

Corollary 5·1— If f is a transcendental meromorphic function and l be an integer $\geqslant 1$, then

$$\Theta_a (\infty, f^{(l)}) \geqslant \frac{l}{l+1} \qquad \qquad (3{\cdot}22)$$

$$\sum_{a \in C} \Theta_a (a, f^{(l)}) \leqslant 1 + \frac{1}{l+1} \qquad \qquad (3{\cdot}23)$$

$$\Theta_a (\infty, f^{(l)}) = \delta_a (\infty, f^{(l)}) \Longleftrightarrow \delta_a (\infty, f^{(l)}) = 1. \qquad \qquad (3{\cdot}24)$$

As a consequence of (3·21) and corollary 4·1 we immediately have the following:

Corollary 5·2—If for each admissible α, $\sum_{a \in C} \delta_\alpha (b, f) = 2$, and there exists $a \in \bar{C}$ such that f-a has no simple zero, then the result of corollary 4·2 holds good.

Further, since $f^{(l)}$ has no simple poles and $f^{(l)}$ has order ρ, we easily find that

Corollary 5.3.—If $l \geqslant 1$ is an integer such that, for each admissible α, $\sum_{a \in C} \delta_a (a, f^{(l)}) = 2$, then the result in corollary 4·2 holds good.

Remark. Since for any admissible α the relation $\delta (a, f) \leqslant \delta_a (a, f) \ \forall a \in \bar{C}\zeta$ is satisfied, one can find that the result of Corollary 5·3 includes as a very special case a result of Ozawa (4) which he has proved for $l = 1$.

4. Lemmas

Lemma 1. Let $a_\mu \in \bar{C}$, $\mu = 1,2 \ldots, q$ $(q > 2)$. Then

$$m_a (r, \infty) + \sum_{\mu=1}^{q} m_a (r, a_\mu) \leqslant 2T_a (r, f) - N_{1,a} (r) + S_a (r) \qquad \qquad (4{\cdot}1)$$

where

$$N_{1,a} (r) = N_a \left(r, \frac{1}{f'} \right) + 2N_a (r, f) - N_a (r, f'), \ N_{1,a} (r) > 0;$$

and

$$S_a (r, f) = m_a \left(r, \frac{f'}{f} \right) + m \left(r, \sum_{\mu=1}^{q} \frac{f'}{(f - a_\mu)} \right) + A(r) \qquad \qquad (4{\cdot}2)$$

$A(r)$ being given by

$$A(r) = \begin{cases} O(r^{-}) & , \alpha > 0 \\ O(\log r) & , \alpha = 0 \end{cases}$$

with modification if $f(0) = \infty$ or $f'(0) = 0$.

PROOF : It follows on dividing by $r^{1+\alpha}$ and proceeding for integration from r_0 to r on both the sides of (*see* Hayman 1964, Theorem 2·1)

$$m(r, \infty) + \sum_{\mu=1}^{q} m(r, a_\mu) \leqslant 2T(r, f) - N_1(r) + S(r).$$

Remark : The above lemma, in a slightly different form, has also been obtained by Toda (1970). He, using an estimate in Theorem 2·2 of Hayman (1964) proved that

$$S_\alpha(r,f) = \begin{cases} O\left(\int_{r_0}^{r} \frac{\log^+ T(t,f)}{t^{1+\alpha}} dt\right), & \text{for } \alpha > 0 \\ O\left((\log r)^2\right) & , \text{for } \alpha = 0 \text{ and } \rho < \infty. \end{cases} \quad (4\cdot3)$$

However, in our present work we need both the estimations of $S_\alpha(r,f)$ as given in (4·2) and (4·3).

Lemma 2—For any admissible α, we have

$$m_\alpha\left(r, \frac{f^{(l)}}{f}\right) = S_\alpha(r,f).$$

PROOF : It follows directly from Lemma 4 of Hayman (1959, p. 13).

Lemma 3 (Toda 1970, lemma 1)—Let $f(z)$ be transcendental. Then

$$\lim_{r\to\infty} \frac{S_\alpha(r,f)}{T_\alpha(r,f)} = 0,$$

Lemma 4— The set of values a, for which $\Theta_\alpha(a,f) > 0$ is countable, and

$$\sum_a [\delta_\alpha(a,f) + \theta_\alpha(a,f) \leqslant \sum_a \Theta_\alpha(a,f) \leqslant 2 \quad \ldots \quad \ldots \quad (4\cdot4)$$

where α is any admissible value.

PROOF : It follows, on the lines of Theorem 2·4 in Hayman (1964), by using Lemmas 1 and 2 above instead of Theorems 2·1 and 2·3 of Hayman (1964). The details are omitted.

The result in Lemma 3, may be rewritten as

$$\sum_{a \in \overline{C}} \Theta_\alpha (a, f) \leqslant 2.$$

Further, the set of values a, for which $\delta_\alpha (a, f) > 0$ is also countable and the inequality

$$\sum_{a \in \overline{C}} \delta_\alpha (a, f) \leqslant 2,$$

holds for any admissible α.

Lemma 5—Let $a_\mu \in \overline{C}$, $\mu = 1, 2 .., q$ $(q > 2)$. Then

$$T_\alpha (r, f^{(l)}) \leqslant S_\alpha (r, f) + m_\alpha (r, f) + (l + 1) N_\alpha (r, f) \quad .. \quad .. \quad (4{\cdot}5)$$

$$T_\alpha (r, f^{(l)}) \geqslant \sum_{\mu=1}^{q} m_\alpha (r, a_\mu) + N_\alpha \left(r, \frac{1}{f^{(l)}}\right) + S_\alpha (r, f) \quad .. \quad .. \quad (4{\cdot}6)$$

for all values of $r \to \infty$, where $S_\alpha(r, f)$ is the same as in Lemma 1 and α being any admissible value.

PROOF : Consider

$$F(z) = \sum_{\mu=1}^{q} \left(\frac{1}{f(z) - a_\mu} \right)$$

therefore, we have

$$\sum_{=1}^{q} m_\alpha (r, a_\mu) \leqslant m_\alpha (r, F f^{(l)}) + m_\alpha \left(r, \frac{1}{f^{(l)}}\right) + 0 (1).$$

But

$$m_\alpha (r, F f^{(l)}) \leqslant m_\alpha (r, F f^{(l-1)}) + m_\alpha \left(r, \frac{f^{(l)}}{f^{(i-1)}}\right)$$

$$\leqslant m_\alpha (r, F f') + \sum_{i=2}^{l-1} m_\alpha \left(r, \frac{f^{(i)}}{f^{(i-1)}}\right)$$

$$\leqslant m_\alpha (r, F f') + (l - 1) S_\alpha (r, f),$$

by Lemma 2 for $l = 1$. Also, in view of (4·2), we have

$$m_\alpha (r, F f^{(l)}) \leqslant - m_\alpha \left(r, \frac{f'}{f}\right) + S_\alpha (r, f) = S \; (r, f).$$

Hence

$$\sum_{\mu=1}^{q} m_a(r, a_\mu) \leqslant m_a\left(r, \frac{1}{f^{(l)}}\right) + S_a(r,f)$$
$$= T_a\left(r, \frac{1}{f^{(l)}}\right) - N_a\left(r, \frac{1}{f^{(l)}}\right) + S_a(r,f)$$
$$= T_a(r, f^{(l)}) - N_a\left(r, \frac{1}{f^{(l)}}\right) + S_a(r,f),$$

which imples the result (4·6). The other part, i.e. (4·5), of the lemma follows by using the facts:

$$m_a(r, f^{(l)}) \leqslant S_a(r,f) + m_a(r,f)$$

and

$$N_a(r, f^{(l)}) \leqslant (l+1) N_a(r,f). \qquad (4\cdot7)$$

5. Proofs of Theorem 1 and its Consequences

Let $<a_i>_1^\infty$ be an infinite sequence of distinct elements of C, which includes every a $\in C$ for which $\delta_a(a,f) > 0$, then

$$\sum_{i=1}^{\infty} \delta_a(a_i, f) = \sum_{a \in C} \delta_a(a,f). \qquad (5\cdot1)$$

Let $q > 2$ be any positive integer, then adding

$\sum_{i=1}^{q} N_a(r, a_i)$ to both sides of (4·6), we have

$$\sum_{i=1}^{q} T_a\left(r, \frac{1}{f-a_i}\right) \leqslant T_a(r, f^{(l)}) + \sum_{i=1}^{q} N_a(r, a_i) - N_a\left(r, \frac{1}{f^{(l)}}\right) - S_a(r,f)$$
$$\leqslant T_a(r, f^{(l)}) + \sum_{i=1}^{q} N_a(r, a_i) - S_a(r,f) \qquad (5\cdot2)$$

Since $N_a\left(r, \frac{1}{f^{(l)}}\right) \geqslant 0$. Also

$$T_a\left(r, \frac{1}{f-a_i}\right) = T_a(r,f) + S_a(r,f)$$

as $r \to \infty$ Therefore

$$q\, T_a(r,f) \leqslant T_a(r,f^{(l)}) + \sum_{i=1}^{q} N_a(r, a_i) - S_a(r,f)$$

which implies

$$q \leqslant \liminf_{r\to\infty} \frac{T_a(r,f^{(l)})}{T_a(r,f)} + \sum_{i=1}^{q} (1 - \delta_a(a_i, f)). \quad \ldots \quad (5\cdot3)$$

Thus

$$\sum_{i=1}^{q} \delta_a(a_i, f) \leqslant \liminf_{r\to\infty} \frac{T_a(r,f^{(l)})}{T_a(r,f)}.$$

Since this holds for all $q > 2$, letting $q \to \infty$ and using (5·1), the first part of (3·1) follows.

Further,

$$\begin{aligned} N_a(r,f^{(l)}) &= N_a(r,f^{(l-1)}) + \bar{N}_a(r,f^{(l-1)}) \\ &= N_a(r,f') + \bar{N}_a(r,f') + \ldots + \bar{N}_a(r,f^{(l-)}) \\ &= N_a(r,f) + \bar{N}_a(r,f) + \bar{N}_a(r,f') + \ldots + \bar{N}_a(r,f^{(l-1)}) \\ &= N_a(r,f) + l\,\bar{N}_a(r.f). \quad \ldots \quad \ldots \quad \ldots \quad (5\cdot4) \end{aligned}$$

Also

$$m_a(r,f^{(l)}) \leqslant m_a\left(r, \frac{f^{(l)}}{f}\right) + m_a(r,f) \leqslant m_a(r,f) + S_a(r,f).$$

Therefore

$$\begin{aligned} T_a(r,f^{(l)}) &\leqslant N_a(r,f) + l\bar{N}_a(r,f) + m_a(r,f) + S_a(r,f) \\ &= T_a(r,f) + l\,\bar{N}_\kappa(r,f) + S_a(r,f), \end{aligned}$$

so that

$$\limsup_{r\to\infty} \frac{T_a(r,f^{(l)})}{T_a(r,f)} \leqslant 1 + l \limsup_{r\to\infty} \frac{\bar{N}_a(r,f)}{T_a(r,f)} = 1 + l(1 - \Theta_a(\infty, f)),$$

and hence the theorem is completely proved.

The proof of Corollary 1·1 is immediate while the proof of Corollary 1·2 follows on dividing both sides of (4·6) by $T_a(r,f)$ and taking the lim inf.

6. Proof of Theorem 2

Using (5·4), we have

$$T_a(r, f^{(l)}) \geqslant N_a(r, f^{(l)}) \geqslant (l+1)\bar{N}_a(r,f). \qquad (6\cdot1)$$

Lemma 1, for $q=2$, $a_1=a$ and $a_2=b$, gives

$$T_a(r,f \leqslant N_a(r,a) + N_a(r,b) + N_a(r,f) - N_a\left(r, \frac{1}{f'}\right) + S_a(r,f)$$

since $2N_a(r,f) - N_a(r,f') \geqslant 0$. Therefore

$$T_a(r,f) \leqslant \bar{N}_a(r,a) + \bar{N}_a(r,b) + N_a(r,f) - N_{0,a}\left(r, \frac{1}{f'}\right) + S(r,f),$$

where $N_{0,a}\left(r, \frac{1}{f'}\right)$ is formed with the zeros of f' which are not zeros of any of $f-a$. and $f-b$. Further, since $N_{0,a}\left(r, \frac{1}{f'}\right) \geqslant 0$, by using the hypothesis, we have

$$T_a(r,f) \leqslant 0(T_a(r,f)) + \bar{N}_a(r,f),$$

for $r \geqslant r_0$. Consequently $T_a(r,f) \sim \bar{N}_a(r,f)$, as $r \to .\infty$ Hence (6·1) implies

$$\liminf_{r\to\infty} \frac{T_a(r, f^{(l)})}{T_a(r,f)} \geqslant l+1. \qquad (6\cdot2)$$

On the other hand, from (4·5), we have

$$T_a(r, f^{(l)}) \leqslant S_a(r,f) + (l+1)T_a(r,f),$$

so that

$$\limsup_{r\to\infty} \frac{T_a(r, f^{(l)})}{T_a(r,f)} \leqslant l+1. \qquad (6\cdot3)$$

Thus the theorem follows from (6·2) and (6·3).

7. Proof of Theorem 3 and its Conesquences

From (5·2), with $l=1$, we have

$$\sum_{i=1}^{q} T_a\left(r, \frac{1}{f-a_i}\right) \leqslant T_a(r,f') + \sum_{i=1}^{q} \bar{N}_a(r,a_i) - N_{0,a}\left(r, \frac{1}{f'}\right) - S_a(r,f),$$

where $N_{0,\alpha}\left(r, \frac{1}{f'}\right)$ is formed with the zeros of f' which are not zeros of any of the $f - a_i$ ($i = 1, 2, \ldots, q$). Since $N_{0,\alpha}\left(r, \frac{1}{f'}\right) \geqslant 0$, this gives

$$\sum_{i=1}^{q} T\left(r, \frac{1}{f-a_i}\right) \leqslant T_\alpha(r, f') + \sum_{i=1}^{q} \bar{N}_\alpha(r, a_i) - S_\alpha(r, f).$$

Now proceeding as in proof of Theorem 1, we arrive at

$$q \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} + \sum_{i=1}^{q} [1 - \Theta_\alpha(a_i, f)]$$

which implies

$$\sum_{i=1}^{q} \Theta_\alpha(a_i, f) \leqslant \liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} .$$

Hence letting $q \to \infty$ and using the fact

$$\sum_{i=1}^{\infty} \Theta_\alpha(a_i, f) = \sum_{a \in C} \Theta_\alpha(a, f),$$

theorem 3 follows.

Proof of corollary 3·1—Under the hypothesis,

$\sum_{a \in C} H_\alpha(a, f) = 2 - H_\alpha(\infty, f)$, (3·4) implies

$$\liminf_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \geqslant 2 - H_\alpha(\infty, f). \quad \ldots \quad \ldots \quad (7\cdot1)$$

On the other hand the second inequality in (3·1), for $l = 1$, gives

$$\limsup_{r\to\infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqslant 2 - H_\alpha(\infty, f). \quad \ldots \quad \ldots \quad (7\cdot2)$$

Hence (7·1) and (7·2) together imply (3·5).

To prove (3·6), let $a \in \bar{C}$. Then consider the sequence $<a_i>_1^\infty$ of distinct elements of $\bar{C}$ which includes every $b \in \bar{C}$ satisfying $b \neq a$ and $\Theta_\alpha(b, f) > 0$, so that

$$\sum_{i=1}^{\infty} \Theta_\alpha(a_i, f) = \sum_{\substack{b \in C \\ b \neq a}} \Theta_\alpha(b, f) = 2 - \Theta_\alpha(a, f). \qquad (7\cdot3)$$

Also, for $q > 2$, the result in lemma 1 may be written as

$$(q-2)\, T_\alpha(r, f) \leqslant \sum_{i=1}^{q} N_\alpha(r, a_i) - N_{1,\alpha}(r) + S_\alpha(r, f)$$

$$\leqslant \sum_{i=1}^{q} N_\alpha(r, a_i) - N_\alpha\left(r, \frac{1}{f'}\right) + S_\alpha(r, f),$$

since $2N_\alpha(r, f) - N_\alpha(r, f') \geqslant 0$. Now, following the arguments as in § 7, we find that

$$(q-2)\, T_\alpha(r, f) \leqslant \sum_{i=1}^{q} N_\alpha(r, a_i) + S_\alpha(r, f).$$

which, on taking $a_q = a$, gives

$$(q-2)\, T_\alpha(r, f) \leqslant \sum_{i=1}^{q-1} \bar{N}_\alpha(r, a_i) + \bar{N}_\alpha(r, a) + S_\alpha(r, f).$$

Consequently

$$\liminf_{r \to \infty} \frac{\bar{N}_\alpha(r, a)}{T_\alpha(r, f)} \geqslant \sum_{i=1}^{q-1} \Theta_\alpha(a_i, f) - 1.$$

Letting $q \to \infty$, we get

$$\liminf_{r \to \infty} \frac{\bar{N}_\alpha(r, a)}{T_\alpha(r, f)} \geqslant \sum_{i=1}^{\infty} \Theta_\alpha(a_i, f) - 1 = 1 - \Theta_\alpha(a, f),$$

by virtue of (7·3). However, by the definition of $H_\alpha(a, f)$, we also note

$$\limsup_{r \to \infty} \frac{\bar{N}_\alpha(r, a)}{T_\alpha(r, f)} = 1 - H_\alpha(a, f).$$

Hence (3·6) follows.

The result (3·7), readily follows from the fact

$$1-H_a(b,f) = \liminf_{r\to\infty} \frac{\bar{N}_a(r,b)}{T_a(r,f)} \leqslant \liminf_{r\to\infty} \frac{N_a(r,b)}{T_a(r,f)}$$
$$= 1-\Delta_a(b,f).$$

Further, if $f-a$ has no simple zero, then $N_a(r,a) \geqslant 2\bar{N}_a(r,a)$. Therefore

$$1-\Delta_a(a,f) \geqslant 2(1-\Theta_a(a,f)$$

which gives

$$2\,\Theta_a(a,f) - \Delta_a(a,f) \geqslant 1.$$

So, if $\Delta_a(a,f) = \Theta_a(a,f)$, then $\Delta_a(a,f) \geqslant 1$ and hence $\Delta_a(a,f) = 1$. This proves (3·8).

Also, since $\bar{N}_a(r,f) = \bar{N}_a(r,f)$, we have

$$\lim \frac{\bar{N}_a(r,f')}{T_a(r,f)} = \lim \frac{\bar{N}_a(r,f)}{T_a(r,f)} \cdot \lim \frac{T_a(r,f)}{T_a(r,f')}$$
$$= \frac{1-\Theta_a(\infty,f)}{2-\Theta_a(\infty,f)} \quad \cdot\cdot \quad (7\cdot4)$$

in view of (3·5) and (3·6). This confirms (3·9).

Finally note

$$N_a(r,f') = N_a(r,f) + \bar{N}_a(r,f)$$
$$= N_a(r,f) + \bar{N}_a(r,f').$$

and then, using (3·5) and (7·4), it can be verified that

$$\limsup_{r\to\infty} \frac{N_a(r,f')}{T_a(r,f')} = \frac{1}{2-\Theta_a(\infty,f)} \limsup_{r\to\infty} \frac{N_a(r,f)}{T_a(r,f)} + \frac{1-\Theta_a(\infty,f)}{2-\Theta_a(\infty,f)}$$

and

$$\liminf_{r\to\infty} \frac{N_a(r,f')}{T_a(r,f')} = \frac{1}{2-\Theta_a(\infty,f)} \liminf_{r\leftarrow 8} \frac{N_a(r,f)}{T_a(r,f)} + \frac{1-\Theta_a(\infty,f)}{2-\Theta_a(\infty,f)}$$

which dispose of the proofs of (3·10) and (3·11) respectively.

Proof of corollary 3·2—Since $\delta_a(a, f) \leqslant \Theta_a(a, f)$ for each $a \in \bar{C}$ and

$\sum_{a \in C} \Theta_a(a, f) \leqslant 2$, it follows that, if $\sum_{a \in C} \delta_a(a, f) = 2$, then

$\sum_{a \in C} \Theta_a(a, f) = 2$ and $\delta_a(a, f) = \Theta_a(a, f)$, for every $a \in \bar{C}$.

Hence (3·12) follows from (3·5), whereas (3·14) from (3·7), (3·10) and (3·11). Also, (3·13) follows by using (3·6) in the following inequalities:

$$\lim_{r\to\infty} \frac{\bar{N}_a(r, a)}{T_a(r, f)} \leqslant \liminf_{r\to\infty} \frac{N_a(r, a)}{T_a(r, a)} \leqslant \limsup_{r\to\infty} \frac{N_a(r, a)}{T_a(r, f)}.$$

Proof of corollary 3.3—It immediately follows by using the hypothesis in (3·9), (3·10) and (3·11), and then proceeding for the method of induction.

8. Proof of Theorem 4

As a consequence of (4·7), we have

$$\limsup_{r\to\infty} \frac{N_a(r, f^{(l)})}{T_a(r, f^{(l)})} \leqslant (l+1) \limsup_{r\to\infty} \left\{ \frac{N_a(r, f)}{T_a(r, f)} \quad \frac{T_a(r, f)}{T_a(r, f^{(l)})} \right\}$$

$$\leqslant \frac{\gamma}{1-\gamma} (l+1),$$

by the left-hand inequality in (3·1) and the hypothesis.

Also, in view of (4·6), we get

$$\frac{N_a\left(r, \frac{1}{f^{(l)}}\right)}{T_a(r, f^{(l)}} \leqslant 1 - \frac{\sum_{i=1}^{q} m_a(r, a_i)}{T_a(r, f^{(l)})} + O(1)$$

$$= 1 - \frac{T_a(r, f)}{T_a(r, f^{(l)})} \sum_{i=1}^{q} \frac{m_a(r, a_i)}{T_a(r, f)} + O(1)$$

which, again in view of the hypothesis and the right hand inequality in (3·1), results in

$$\limsup_{r\to\infty} \frac{N_a\left(r, \frac{1}{f^{(l)}}\right)}{T_a(r, f^{(l)})} \leqslant \frac{\gamma}{1+\gamma l} (l+1).$$

This completes the proof of the theorem.

9. Proofs of Theorem 5 and Corollary 5·1

Since no zero of f-a is of order less then p, we have

$$N_\alpha(r,a) \geqslant p\,\bar{N}_\alpha(r,a),$$

and so

$$\limsup_{r\to\infty} \frac{N_\alpha(r,a)}{T_\alpha(r,f)} \geqslant p \limsup_{r\to\infty} \frac{\bar{N}_\alpha(r,a)}{T_\alpha(r,f)} .$$

Hence

$$\delta_\alpha(a,f) \leqslant 1 - p(1 - \Theta_\alpha(a,f)).$$

This proves (3·20) and the proof of (3·21) is immediate.

Proof of corollary 5·1—Since $f^{(l)}$ has no pole of order less than $l+1$, by taking $f^{(l)}$ in place of f and $p = l+1$ in (3·20), we observe

$$\delta_\alpha(\infty, f^{(l)}) \leqslant 1 - (l+1)\,[1 - \Theta_\alpha(\infty, f^{(l)})],$$

which implies (3·22) for $\delta_\alpha(\infty, f^{(l)}) \geqslant 0$, and (3·23) follows from the fact that $\sum_{a \in C} \Theta_\alpha(a, f^{(l)}) \leqslant 2$, whereas, (3·24) is an easy consequence of (3·21).

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