COMBINED FREE AND FORCED CONVECTION FLOW OF AN ELASTICO-VISCOUS LIQUID PAST A HOT POROUS CIRCULAR CYLINDER

by SHANKAR PRASAD MISHRA, *Post Graduate Department of Mathematics, Utkal University, Bhubaneswar*

and

UMAKANTA MOHAPATRA, *Department of Mathematics, Ravenshaw College, Cuttack*

(Communicated by J. N. Kapur, F.N.A.)

(*Received 29 September 1973* ; *after revision 6 February 1974*)

In this paper the authors study the flow of an elastico-viscous liquid past a hot vertical porous circular cylinder by including the free convection effects parallel to the generators of the cylinder. The flow phenomenon has been characterized by the non-dimensional parameters R (suction parameter), R_c (elastic parameter), G (Grashof number), and σ (Prandtl number) and the effects of these parameters on the velocity and temperature distributions, shearing stress at the surface of the cylinder are studied.

1. INTRODUCTION

Lew (1956) pioneered the investigation on the effect of transverse curvature to the steady flow of a viscous liquid past a porous circular cylinder. He has shown that the growth of boundary layer is inhibited by the application of a radial suction to the cylinder and a steady asymptotic solution for the axial velocity is possible. Mishra (1963) and Sinha Roy (1966) have solved Lew's problem replacing viscous liquid by an elastico-viscous liquid. In this note we extend Sinha Roy's analysis to include free convection effects parallel to the generators of a vertical porous circular cylinder. The elastico-viscous liquid model considered in this note consists of the equations :

$$p_{ik} = -pg_{ik} + p'_{ik} \tag{1}$$

$$p'^{ik} = 2\eta_0 e^{ik} - 2k_0 \left[\frac{\partial e^{ik}}{\partial t} + v^m e^{ik}_{,m} - v^k{}_{,m} e^{im} - v^i{}_{,m} e^{mk} \right] \tag{2}$$

where

$$2e_{ik} = v_{i,k} + v_{k,i}$$

v^i is the velocity vector, p^{ik} the stress tensor, p the mean pressure and g_{ik} the metric tensor of a fixed coordinate system. The limiting viscosity at small rates of shear is

$$\eta_0 = \int_0^\infty N(\tau)\, d\tau$$

and

$$k_0 = \int_0^\infty \tau N(\tau)\, d\tau$$

is the short memory coefficient, $N(\tau)$ is the relaxation spectrum as introduced by Walters (1960). A detailed description of the model has been given in a paper by Beard and Walters (1964). The equations of motion, continuity and energy are

$$\rho \left[\frac{\partial v^i}{\partial t} + v^j v^i{}_{,j} \right] = -p_{,i} + p'^{ij}{}_{;j} + F_i \tag{3}$$

$$v_{i,i} = 0,$$

and

$$\rho c \frac{DT}{Dt} = k \nabla^2 T + \phi \tag{5}$$

where ρ is the density, F_i the body force in the ith direction, T the temperature, k the conductivity, c the specific heat and ϕ the dissipation function given by

$$\phi = p'_{ij}\, e_{ij}.$$

2. Formulation of the Problem

We work through cylindrical polar coordinate system (r, θ, z) with z-axis along the axis of the cylinder which is vertical and is of radius a. It is kept at a temperature T_w. Since we study the axial symmetry flow at a long distance from the leading edge, the θ and z coordinates will not appear in discussion. The velocity field may be taken in the from

$$v^r = u(r), \quad v^\theta = 0, \quad v^z = w(r). \tag{6}$$

The velocity field (6) is compatible with the continuity condition (4) if

$$u = \frac{au_a}{r}$$

where u_a is the constant normal velocity at the surface of the wall. The non-vanishing stress components from (2) are

$$p'^{rr} = -\frac{2\eta_0 a u_0}{r^2}, \quad p'^{zz} = -2k_0 \left(\frac{dw}{dr}\right)^2 \tag{8}$$

$$p'^{rz} = \eta_0 \frac{dw}{dr} - \frac{k_0\, a\, u_a}{r} \left[\frac{d^2w}{dr^2} + \frac{3}{r}\frac{dw}{dr} \right] \tag{9}$$

$$p'^{\theta\theta} = \frac{2\eta_0\, a\, u_a}{r^2} + \frac{8k_0\, a^2\, u_a^2}{r^4}. \tag{10}$$

The momentum equations from (3) are

$$\rho u \frac{du}{dr} = -\frac{\partial p}{\partial r} + \frac{dp'^{rr}}{dr} + \frac{p'^{rr} - p'^{\theta\theta}}{r} \tag{11}$$

$$\rho u \frac{dw}{dr} = \frac{dp'^{rz}}{dr} + \frac{1}{r} p'^{rz} + \rho g\beta(T - T_\infty) \tag{12}$$

where T_∞ is the free stream temperature, g is the acceleration due to gravity and β the coefficient of thermal expansion. In eqn. (12) the term $g\beta(T - T_\infty)$ represents the buoyancy force arising out of temperature variations in the fluid and the density variation is taken into account while writing down this force only.

From eqns. (7)--(12), we get

$$\frac{dp}{d\eta} = \frac{1}{\eta^3} - \frac{8R_c}{\eta^5} \tag{13}$$

$$\bar{w}'' + \frac{1 - R_c}{\eta} \bar{w}' - R_0 R_c \left[\frac{1}{\eta} \bar{w}''' + \frac{3}{\eta^2} \bar{w}'' - \frac{3}{\eta^3} \bar{w}' \right] = -G\theta \tag{14}$$

where

$$\left. \begin{aligned} r &= a\eta, \quad \bar{w} = w/W, \quad R_c = \frac{k_0}{\rho a^2}, \quad \sigma = \frac{c\eta_0}{k} \\ \theta &= \frac{T - T_\infty}{T_w - T_8}, \quad G = \frac{g\beta a^2 (T - T_\infty)\rho}{\eta_0 W} \\ R_0 &= \frac{\rho a u_a}{\eta_0}, \quad P \frac{p}{\rho u_a{}^2}. \end{aligned} \right\} \tag{15}$$

The heat energy equation (5) can be written in the form

$$\theta'' + \frac{1 - \sigma R_0}{\eta} \theta' = -4\sigma E \left[\frac{1}{\eta^4} + \frac{2 R_0 R_c}{\eta^6} \right]$$

$$- \frac{R_1^2}{R_0^2} E\sigma \left[\bar{w}'^2 - \frac{R_0 R_c}{\eta} \bar{w}' \left\{ \bar{w}'' + \frac{3}{\eta} \bar{w}' \right\} \right]. \tag{16}$$

The boundary conditions on the velocity and temperature distributions are

$$\eta = 1 : \quad \bar{w} = 0, \quad \theta = 1 \quad \text{and} \quad \eta \to \infty : \quad \bar{w} \to 1, \quad \theta \to 0. \tag{17}$$

σE is assumed to be small enough to be neglected in this problem. This is same as neglecting the dissipation function which is justified for slow motion as is the case in free convection flow. Hence eqn. (16) can be written as

$$\theta'' + \frac{1 - \sigma R_0}{\eta} \theta' = 0$$

which when integrated with the bounadry conditions (17) gives

$$\theta = \eta^{-\sigma R} \tag{18}$$

where $R_0 = -R$ and $R>0$. It is to be noted that a solution is possible only if R_0 is negative.

Integration of (13) gives

$$p_\infty - p = \frac{1}{2\eta^2} - \frac{2R_c}{\eta^4}$$

where p_∞ is the pressure at infinity. This shows that the radial pressure variation $(p_\infty - p)$ decreases as R_c increases and is independent of the suction parameter R.

In view of eqns. (14) and (18), we get by omitting bars over $\bar{w}$,

$$w'' + \frac{1+R}{\eta} w' + RR_c \left[\frac{w'''}{\eta} + \frac{3w''}{\eta^2} - \frac{3w'}{\eta^3} \right] = -G\eta^{-\sigma R}. \tag{19}$$

where dashes denote differentiation with respect to η.

3. Solution of Equation

It is difficult to obtain a closed form solution of (19). So we adopt an iteration method to solve this. We define a sequence of functions $w_0, w_1, w_2, \cdots\cdots$ (following Collatz (1959)) as solutions of equations given by

$$w_n'' + \frac{1+R}{\eta} w_n' + RR_c \left[\frac{1}{\eta} w_{n-1}''' + \frac{3}{\eta^2} w_{n-1}'' - \frac{3}{\eta^3} w_{n-1}' \right] = -G\eta^{-\sigma R} . \tag{20}$$

The boundary conditions are

$$\left.\begin{array}{l} \eta = 1 : w_n = 0, \quad w_{-1} = w_{-1}' = w_{-1}'' = w_{-1}''' = 0 \\ \eta \to \infty : \quad w_n \to 1. \end{array}\right\} \tag{21}$$

The zeroth order iterate is

$$w_0 = 1 + (A_1 - 1)\,\eta^{-R} - A_1\,\eta^{(2-\sigma R)} \tag{22}$$

where $\sigma R > 2$. It is to be noted that besides the condition that R is positive, we must have the condition $\sigma R > 2$ for the existence of a solution.

The first order iterate is

$$w_1 = 1 + [A_1 - 1 - R_c R^2 + (A_2 + A_3)]\,\eta^{-R} - A_1\,\eta^{2-\sigma R}$$
$$- R^2 R_c\,[\,A_2\,\eta^{-(R+2)} + A_3\,\eta^{-\sigma R}]$$

The second order iterate is

$$w_2 = 1 + A_7\,\eta^{-R} - [A_5\,\eta^{-(R+2)} + A_6\,\eta^{-\sigma R} + A_1\,\eta^{2-R\sigma}] + o(R^2{}_c) \tag{24}$$

where

$$A_1=\frac{G}{(2-\sigma R)(2-\sigma R+R)}, \qquad A=\tfrac{1}{2}(A_1-1)(2-R),$$

$$A_3=\frac{A_1(2-\sigma R)(4-\sigma R)}{R^2(\sigma-1)}, \qquad A_4=A_1-1-R^2R_c(A_2+A_3)$$

$$A_5=\tfrac{1}{2}R^2R_c(2-R)(A_1-1), \; A_6=\frac{R_c(2-\sigma R)(4-\sigma R)A_1}{\sigma-1}$$

$$A_7=-1+A_1+A_5+A_6.$$

The shearing stress ζ_0 at the surface of the cylinder ($\eta=1$) is from (9), (15) and (24) is

$$\tau_0=(1+3RR_c)[(R+2)A_5+\sigma RA_6-(2-\sigma R)A_1-RA_7]$$
$$-RR_c[(2-\sigma R)(1-\sigma R)A_1-R(1+R)(A_1-1)] \qquad (25)$$

since for an asymptotic solution $\sigma R>2$, eqn. (24) will hold good if $2-\sigma R+R\neq 0$ and this is surely the case if $\sigma<1$.

If the free stream velocity is opposite to the direction of buoyancy force, we have to seek a solution of (19) satisfying

$$w=0 \text{ at } \eta=1 \text{ and } w\to -1 \text{ as } \eta\to\infty$$

and the second iterate in this case is

$$w_2=-1+B_6\eta^{-R}-[B_5\,\eta^{-(R+2)}+A_1\,\eta^{2-\sigma R}]+o(R_c^2)$$

where

$$B_2=\tfrac{1}{2}(A_1+1)(2-R)$$
$$B_4=1+A_1+R^2R_c)(B_2+A_3)$$
$$B_5=\tfrac{1}{2}R^2\,R_c\,(2-R))(1+A_1)$$
$$B_6=B_5+A_1+A_6+1.$$

We can in this case write the expression for the shearing stress at the wall.

4. Discussion Of Results

For convenience a normalized distant variable

$$\lambda=\frac{\eta-1}{\eta}$$

has been introduced such that when

$$\eta=1:\lambda=0 \text{ and when } \eta\to\infty:\lambda\to 1.$$

Table I which represents the computed values of (w_1-w_0) and (w_2-w_1) for fixed values of R_c and R and for diffierent values of λ shows that the iteration is very rapidly convergent and the second iteration gives a very accurate result. The technique adopted, to study convergence in a narrow sense is the Cauchy's principle of convergence.

TABLE I

Table showing the convergence of iteration

$R=4,\ R_c=0.005,\ \sigma=0.6,\ G=10$

η	w_0	w_1	w_2	w_1-w_0	w_2-w_1
0.05	0.19789	0.20989	0.21367	0.01200	0.00378
0.10	0.35798	0.37410	0.37848	0.01612	0.00438
0.15	0.49046	0.50733	0.51134	0 01687	0.00404
0.20	0.60055	0.61655	0.61993	0.01600	0.00338
0.25	0.69153	0.70583	0.70855	0.01430	0.0072
0.30	0.76595	0.77816	0.78028	0.01221	0.00212
0.35	0.82599	0.83602	0.83761	0.01003	0.00159
0.40	0.87367	0.88157	0.88273	0.00790	0.00116
0.45	0.91080	0.91679	0.91760	0.00599	0.00081
0.50	0.93907	0.94342	0.94397	0.00435	0.00055
0.55	0.96002	0.96303	0.96338	0.00301	0·00035
0.60	0.97540	0.97700	0.97722	0.00196	0.00022
0.65	0.98537	0.98656	0.98669	0.00119	0.00013
0.70	0.99210	0.99276	0.99283	0.00066	0.00007
0.75	0.99619	0.99652	0.99655	0.00033	0.00003
0.80	0.99844	0.99857	0.99859	0.00013	0.00002
0.85	0.99950	0.99955	0.99955	0.00005	0.00000
0.90	0.99990	0.99991	0.99991	0.00001	.000000
0.95	0.99999	0.99999	0.99999	0.00000	.000000
1.00	1.00000	1.00000	1.00000	0.00000	.000000

Figure 1 shows that in a thin liquid layer near the cylinder the elasticity of the liquid increases the velocity at any point and at far away from it the velocity tends to be constant.

Figure 2 shows that in a thin liquid layer near the cylinder velocity increases with the Grashof number G. This seems physically plausible as increase in G increases the buoyant force leading to an increase in the velocity.

Figure 3 shows that in a thin liquid layer near the cylinder the velocity increases with R, but at far away from the cylinder an opposites effect is observed.

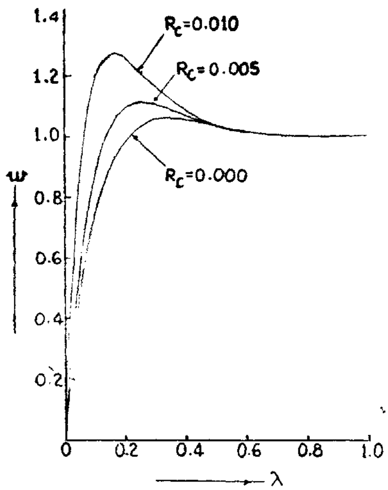

Fig. 1. Velocity distribution for different values of the elastic number.

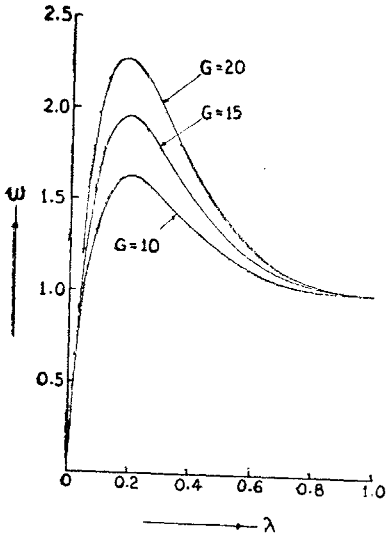

Fig. 2. Velocity distribution for different values of the Grashof number.

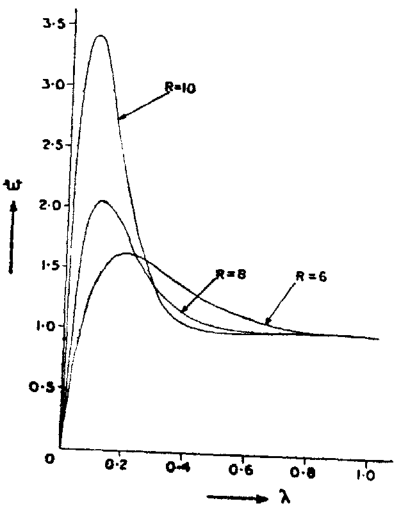

Fig. 3. Velocity distributian for different values of the suction parameter.

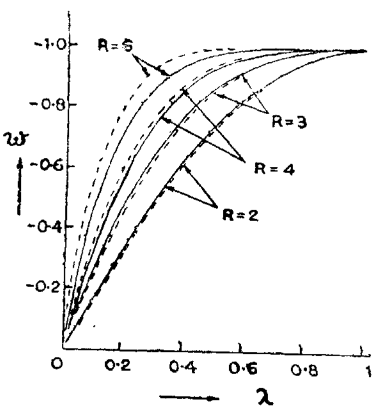

Fig. 4. Velocity distribution for different values of suction parameter and elastic number when the free stream velocity is anti-parallel to the buoyancy force.

Figure 4 is drawn for the case when the free stream velocity is antiparallel to the buoyancy force. This figure shows that the velocity decreases as R

increases. When the suction parameter $R > 3$, the elasticity of the liquid decreases the velocity, whereas for $R<3$ an opposite effect is observed. The elastic effect in the liquid is more pronounced for higher values of R.

Table II shows that the shearing stress at the surface of the cylinder decreases as the elasticity of the liquid increases. The effect of the suction velocity and the Grashof number is to increase the shearing stress at the surface of the cylinder.

TABLE II

Shearing Stress at the surface of the cylinder

$\sigma = 0.75$

R_c \ R	6	8	10	G
	8.00	9.25000	10.9090	5
0.0	10.000	10.50000	11.81818	10
	12.000	12.71000	12.72727	12
	7.99999	9.25000	10.90908	5
0.1	9.99997	10.50000	11.81819	10
	11.99997	12.12000	12.72726	15
	7.99999	9.20000	10.90908	5
0.2	9.99995	10.25000	11.81816	10
	11.99995	11.75000	12.72726	15

REFERENCES

Beard, D. W., and Walters, Ken (1967). Elastico-viscous boundary-layer flows. *Proc. Camb. Phil. Soc.*, **60**, 667.

Collatz, L. (1959). The Neumerical Treatment of Differential Equations. Springer Verlag, New York, P. 195.

Lew, H. G. (1956). The asymptotic behaviour of the boundary layer to transverse curvature. *J. aerospace Sci.* **23**, 267—277.

Mishra, S. P. (1963). Elastico-vicious fluid flow past a circular cylinder or a flat plate with Suction *Ganita*, **14**, 99—102.

Sinha Roy, J. (1966). Elastico-viscous fluid flow past porous circular cylinder, *AIAA Jl*, **4**, 1443—1445.

Walters, K. (1960). The motion of an elastico-viscous liquid contained between two coaxial cylinders, *Jl. Mech. appl. Math.*, **13** 444—461.