LARGE DEFLECTION OF HEATED EQUILATERAL TRIANGULAR PLATE

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Following Berger's (1955) method, the large deflections of heated equilateral triangular plates subject to uniform load have been investigated with the help of trilinear coordinates. The deflections have been obtained for stationary and non-stationary temperature distributions.

NOMENCLATURE

The following nomenclature is used in this paper :

$D = \dfrac{Eh^3}{12(1-\sigma^2)}$ = flexural rigidity of the plate,

$\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2} = \dfrac{\partial^2}{\partial p_1^2}+\dfrac{\partial^2}{\partial p_2^2}+\dfrac{\partial^2}{\partial p_3^2}-\dfrac{\partial^2}{\partial p_1 \partial p_2}-\dfrac{\partial^2}{\partial p_2 \partial p_3}-\dfrac{\partial^2}{\partial p_3 \partial p_1}$

= Laplacian operator in trilinear co-ordinates,

q = uniform load,

E, σ, α = Young's modulus, Poisson's ratio and coefficient of thermal expansion respectively,

u, v = displacements corresponding to the x and y axes,

w = normal displacement,

$e_{xx} = \dfrac{\partial u}{\partial x}+\frac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2,$

$e_{yy} = \dfrac{\partial v}{\partial y}+\frac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2,$

$e = e_{xx}+e_{yy}$ = first invariant,

$e_2 = e_{xx}e_{yy}-\frac{1}{4}e_{xy}^2$ = second invariant,

χ = thermal diffusivity,

h = thickness of the plate,

INTRODUCTION

The classical large deflection plate problems usually lead to non-linear differential equations which cannot be exactly solved. Berger (1955) has shown that if, in deriving the differential equations from the expression for strain energy, the strain energy due to second invariant in the middle plane of the plate is neglected, a simple fourth order differential equation coupled with a non-linear second order equation is obtained. Although no complete explanation of the method is set forth, the stresses and deflections obtained by Berger for rectangular and circular plates agree well with those found from more precise analysis. This approximate method has been extended to orthotropic plates by Iwinski and Nowinski (1957) and further boundary value problems associated with circular and rectangular plates have been solved by Nowinski (1958). The above technique of Berger has been used quite elegantly by Thein Wah and Robert Schmidt (1963) and Nash and Modeer (1959) to obtain satisfactory results.

Basuli (1968) has extended this approximate method of Berger to problems under uniform load and heating under stationary temperature distribution.

In this paper the author has applied this method devised by Berger and Basuli to investigate the large deflections of heated equilateral triangular plates with the help of trilinear coordinates.

COORDINATES USED

Trilinear Coordinates (*vide Sen 1968*)

Let ABC be an equilateral triangular plate. The centroid O is taken as origin, the x-axis perpendicular to the side BC, and the y-axis parallel to this side. If (x, y) be the coordinates of a point P within the triangle, p_1, p_2, p_3, the three perpendiculars from P on CA, AB and BC respectively, $2a$ the length of each side of the triangle and r the radius of the inscribed circle, then one gets

$$p_1 = r + \frac{x}{2} - \frac{y\sqrt{3}}{2}$$

$$p_2 = r + \frac{x}{2} + \frac{y\sqrt{3}}{2}$$

$$p_3 = r - x$$

$p_1+p_2+p_3 = a\sqrt{3} = \text{constant} = k$ (say).

Also,

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial p_1^2}+\frac{\partial^2}{\partial p_2^2}+\frac{\partial^2}{\partial p_3^2}-\frac{\partial^2}{\partial p_1 \partial p_2}-\frac{\partial^2}{\partial p_2 \partial p_3}-\frac{\partial^2}{\partial p_3 \partial p_1}.$$

THEORY

Combining the strain energy due to bending and stretching of the middle surface of the plate loaded normally without temperature and the strain energy due to heating, the total potential energy, V, is given by [vide Berger 1955, Boley and Weiner 1960)

$$V=\iint_S \left[\frac{D}{2}\left\{(\nabla^2 w)^2+\frac{12}{h^2}e^2-2(1-\sigma)\left[\frac{12}{h^2}e_2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\right.$$

$$\left.-\,qw\right]dxdy-\iint_S\int_{-h/2}^{h/2}\frac{E\alpha}{1-\sigma}\,T'(x,y,z)\,[\,e-z\nabla^2 w\,]\,dx\,dy\,dz \tag{1}$$

where the symbol $\iint_S$ indicates integration over the surface S of the plate.

Let the temperature distribution $T'(x, y, z)$ be assumed in the form (vide Basuli 1958)

$$T'(x,y,z)=T_0(x,y)+g(z)\,T(x,y) \tag{2}$$

where (i) $\int_{-h/2}^{h/2} zg(z)\,dz=f(h)$ and (ii) $\int_{-h/2}^{h/2} g(z)\,dz=0.$ (3)

Combining (1), (2) and (3) one gets

$$V=\iint_S\left[\frac{D}{2}\left\{(\nabla^2 w)^2+\frac{12}{h^2}e^2-2(1-\sigma)\left[\frac{12}{h^2}e_2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}\right.$$

$$\left.-qw-\frac{E\alpha}{1-\sigma}\left\{T_0 eh-f(h).\ T.\ \nabla^2 w\right\}\right]dxdy. \tag{4}$$

Neglecting e_2 and using Eulers's variational equations, the following two differential equations are obtained (vide Basuli 1968)

$$\nabla^2(\nabla^2-\beta_1^2)w=\frac{1}{D}\left[q-\frac{E\alpha f(h)}{1-\sigma}\nabla^2 T\right] \tag{5}$$

$$e-(1+\sigma)\,\alpha T_0=\frac{\beta_1^2 h^2}{12} \tag{6}$$

where β_1^2 is the normalized constant of integration.

ANALYSIS

Case I: Stationary Temperature Distribution

If there is no source of heat inside the plate the following differential equations must be satisfied for stationary temperature distribution (vide Nowacki 1962, p. 439)

$$\nabla^2 T_0-\epsilon T_0=-\frac{\epsilon_0}{2}(\theta_1+\theta_2) \tag{7}$$

$$\nabla^2 T-\frac{12}{h^2}(1+\epsilon)\,T=-\frac{12\epsilon}{h^3}(\theta_1-\theta_2) \tag{8}$$

where θ_1, θ_2 denote temperatures at the upper and lower media of the plate respectively.

Let T_0=constant, $T_1 \neq$constant, but $\theta_1-\theta_2$=constant.

From (8)

$$\nabla^2 T - l^2 T = -\beta \tag{9}$$

where

$$\beta = \frac{12\epsilon}{h^3}(\theta_1-\theta_2), \quad l^2 = \frac{12}{h^2}(1+\epsilon).$$

Let β be expressed in the following form of the Fourier series

$$\beta = \sum_{n=1}^{\infty} a_n \left(\sin \frac{2n\pi p_1}{k} + \sin \frac{2n\pi p_2}{k} + \sin \frac{2n\pi p_3}{k} \right) \tag{10}$$

where $a_n = \frac{2\beta}{n\pi}$ (vide Sen 1968).

Also T can be expressed in the form of the similar series

$$T = \sum_{n=1}^{\infty} b_n \left(\sin \frac{2n\pi p_1}{k} + \sin \frac{2n\pi p_2}{k} + \sin \frac{2n\pi p_3}{k} \right) \text{ where } b_n \text{ is a constant.} \tag{11}$$

The above form of T satisfies the following boundary conditions:

$$T = 0 \quad \text{on } p_1 = 0, \ p_2 = 0 \text{ and } p_3 = 0.$$

Combining (9), (10) and (11) one gets

$$b_n = \frac{2\beta}{n\pi \left(\frac{4n^2\pi^2}{k^2} + l^2 \right)}. \tag{12}$$

Thus T is determined.

To solve the differential equation (5), w is assumed in the form

$$w = \sum_{n=1}^{\infty} A_n \left(\sin \frac{2n\pi p_1}{k} + \sin \frac{2n\pi p_2}{k} + \sin \frac{2n\pi p_3}{k} \right) \tag{13}$$

where A_n is a constant.

The above form of w satisfies the following boundary conditions for simply-supported edges:

$$w=0, \ \nabla^2 w=0 \quad \text{at } p_1=0, \ p_2=0 \text{ and } p_3=0.$$

Also expanding the uniform load q by the following form of the Fourier series

$$q = \sum_{n=1}^{\infty} q_n \left(\sin \frac{2n\pi p_1}{k} + \sin \frac{2n\pi p_2}{k} + \sin \frac{2n\pi p_3}{k} \right) \tag{14}$$

where $q_n = \frac{2q}{n\pi}$

and substituting (10), (11), (12), (13) and (14) in (5) A_n is obtained. Putting this value of A_n in (13) the deflection w is given by

$$w=\sum_{n=1}^{\infty}\frac{\dfrac{2q}{Dn\pi}+\dfrac{2E\alpha f(h)\beta.\,4n^2\pi^2}{D(1-\sigma)n\pi\left(\dfrac{4n^2\pi^2}{k^2}+l^2\right)k^2}}{\dfrac{4n^2\pi^2}{k^2}\left(\dfrac{4n^2\pi^2}{k^2}+\beta_1^2\right)}\times$$

$$\times\left(\sin\frac{2n\pi p_1}{k}+\sin\frac{2n\pi p_2}{k}+\sin\frac{2n\pi p_3}{k}\right). \tag{15}$$

The boundary conditions of u and v are

$$\left.\begin{array}{lll} u=0 & \text{at} & p_3=0 \\ \sqrt{3}v+u=0 & \text{at} & p_2=0 \\ \sqrt{3}v-u=0 & \text{at} & p_1=0. \end{array}\right\} \tag{16}$$

The above boundary conditions of u and v can be satisfied by the configurations of the forms :

$$u=\sum_{m=1}^{\infty}\sqrt{3}\,B_m\left\{\sin\frac{2m\pi(p_2+p_3)}{k}+\sin\frac{2m\pi(p_1+p_3)}{k}\right\}$$

$$v=\sum_{m=1}^{\infty}B_m\left\{\sin\frac{2m\pi(p_1+p_3)}{k}-\sin\frac{2m\pi(p_2+p_3)}{k}\right\} \tag{17}$$

where B_m is a constant.

Substituting the values of u, v and w from (17) and (13) in equation (6) and integrating over the area of the plate, one gets the following equation determining the constant β_1^2

$$\sum_{n=1}^{\infty}\frac{3n^2\pi^2}{k}\left[\frac{\left\{\dfrac{2q}{Dn\pi}+\dfrac{2E\alpha f(h).\,\beta.\,4n^2\pi^2}{k^2D(1-\sigma)n\pi\left(\dfrac{4n^2\pi^2}{k^2}+b^2\right)}\right\}}{\dfrac{4n^2\pi^2}{k^2}\left(\dfrac{4n^2\pi^2}{k^2}+\beta_1^2\right)}\right]-(1+\sigma)\alpha T_0=\frac{\beta_1^2h^2}{12}. \tag{18}$$

As $\beta_1\to0$ and $q\to0$ (15) reduces to [when $g(z)=z$]

$$w=\sum_{n=1}^{\infty}\frac{3\beta\alpha(1+\sigma)a^2}{2n^3\pi^3\left(\dfrac{4n^2\pi^2}{3a^2}+l^2\right)}\times\left(\sin\frac{2n\pi p_1}{k}+\sin\frac{2n\pi p_2}{k}+\sin\frac{2n\pi p_3}{k}\right) \tag{19}$$

which is the corresponding small deflection for a simply-supported plate in the absence of any load.

Case II : Non-Stationary Quasi-Static Temperature Distribution

In this case the temperature distribution is assumed in the form

$$T'(x,y,z\,;t)=T_0(x,y\,;t)+g(z)\,T(x,y\,;t). \tag{20}$$

If there is no source of heat inside the plate, the temperature distribution in the plate must satisfy the following equations (vide Nowacki, 1962, p. 494)

$$\nabla^2 T_0 \frac{T_0}{\chi} - \epsilon_0 T_0 = -\frac{\epsilon_0}{2}(\theta_1+\theta_2) \tag{21}$$

$$\nabla^2 T - \frac{1}{\chi}\frac{\partial T}{\partial t} - \frac{12}{h^2}(1+\epsilon)T = -\frac{12\epsilon}{h^3}(\theta_1-\theta_2) \tag{22}$$

where θ_1 and θ_2, temperatures at the upper and lower media of the plate respectively, have opposite signs and $\theta_1+\theta_2$ is a small quantity, time being a parameter only.

Let T_0 = constant, $T_1 \neq$ constant and $\theta_1-\theta_2 = \theta_0\,\eta(t)$ where $\eta(t)$ is the Heaviside unit function.

The following boundary conditions on T are imposed :

$$T = 0 \text{ on } p_1 = 0,\ p_2 = 0,\ p_3 = 0 \text{ and } T = 0 \text{ at } t = 0.$$

Applying Laplace Transform in eqn. (22) one gets

$$\nabla^2 \bar{T} - \left(\frac{p}{\chi}+c^2\right)\bar{T} = -\frac{s^2\theta_0}{p} \tag{23}$$

where $\bar{T} = \int_0^\infty T e^{-pt}\,dt,\quad c^2 = \frac{12}{h^2}(1+\epsilon)$ and $s^2 = \frac{12\epsilon}{h^3}$.

Let
$$\bar{T} = \sum_{n=1}^{\infty} c_n(p)\left(\sin\frac{2n\pi p_1}{k} + \sin\frac{2n\pi p_2}{k} + \sin\frac{2n\pi p_3}{k}\right) \tag{24}$$

Also
$$\frac{s^2\theta_0}{p} = \sum_{n=1}^{\infty}\frac{2s^2\theta_0}{np}\left(\sin\frac{2n\pi p_1}{k} + \sin\frac{2n\pi p_2}{k} + \sin\frac{2n\pi p_3}{k}\right). \tag{25}$$

Combining (23), (24) and (25) one gets

$$C_n(p) = \frac{2s^2\theta_0}{n\pi p\left(\frac{4n^2\pi^2}{k^2}+\frac{p}{\chi}+c^2\right)}. \tag{26}$$

Combining (24) and (26) and by inverse Laplace transform one gets

$$T = \sum_{n=1}^{\infty}\frac{2s^2\theta_0}{n\pi\left(\frac{4n^2\pi^2}{k^2}+c^2\right)}\left\{1-e^{-\left(\frac{4n^2\pi^2}{k^2}+c^2\right)\chi t}\right\} \times\left(\sin\frac{2n\pi p_1}{k} + \sin\frac{2n\pi p_2}{k} + \sin\frac{2n\pi p_3}{k}\right). \tag{27}$$

Assuming the same expressions for w and q given by (13) and (14) and substituting these and (27) in eqn. (5) the deflection w is given by

$$w = \sum_{n=1}^{\infty} \left[\frac{\dfrac{2q}{Dn\pi} + \dfrac{2s^2\theta_0}{Dn\pi\left(\dfrac{4n^2\pi^2}{k^2}+c^2\right)}\left\{1 - e^{-\left(\frac{4n^2\pi^2}{k^2}-c^2\right)\chi t}\right\} \cdot \dfrac{E\alpha f(h)}{1-\sigma}\dfrac{4n^2\pi^2}{k^2}}{\dfrac{4n^2\pi^2}{k^2}\cdot\left(\dfrac{4n^2\pi^2}{k^2}+\beta_1^2\right)} \times \right.$$

$$\left. \times\left(\sin\frac{2n\pi p_1}{k} + \sin\frac{2n\pi p_2}{k} + \frac{2n\pi p_3}{k}\right)\right]. \tag{28}$$

To determine the constant β_1^2 the same expressions for u and v given by (17) are assumed. Substituting these values of u, v and w from eqn. (28) in eqn. (6) and integrating over the area of the plate one gets

$$\sum_{n=1}^{\infty} \frac{3A_n^2\, n^2\pi^2}{k^2} - (1+\sigma)\,\alpha T_0 = \frac{\beta_1^2 h^2}{12} \tag{29}$$

where

$$A_n = \frac{\dfrac{2q}{Dn\pi} + \dfrac{2s^2\theta_0}{Dn\pi\left(\dfrac{4n^2\pi^2}{k^2}+c^2\right)}\left\{1 - e^{-\left(\frac{4n^2\pi^2}{k^2}+c^2\right)\chi t}\right\} \cdot \dfrac{E\alpha f(h)}{1-\sigma}\cdot\dfrac{4n^2\pi^2}{k^2}}{\dfrac{4n^2\pi^2}{k^2}\left(\dfrac{4n^2\pi^2}{k^2}+\beta_1^2\right)}\,. \tag{30}$$

As $q\to 0$, $\beta_1\to 0$, and $g(z)=z$ eqn. (28) reduces to

$$w = \sum_{n=1}^{\infty} \frac{3s^2\,\theta_0\, a^2(1+\sigma)\alpha}{\alpha n^3\pi^3\left(\dfrac{4n^2\pi^2}{3a^2}+c^2\right)}\left\{1 - e^{-\left(\frac{4n^2\pi^2}{3a^2}+c^2\right)\chi t}\right\} \times$$

$$\times\left(\sin\frac{2n\pi p_1}{k} + \sin\frac{2n\pi p_2}{k} + \sin\frac{2n\pi p_3}{k}\right) \tag{31}$$

which is the corresponding small deflection for a simply-supported plate in the absence of any load.

Numerical Calculation

For the case of non-stationary temperature distribution, the deflections are calculated at the origin for various values of χt considering the following data :

$$q=0,\quad a=20,\quad h=2,\quad \epsilon=0.05,\quad \alpha=1.2\times 10^{-5},\quad \sigma=0.3,\quad T_0=10^{\circ}. \tag{32}$$

At the origin $p_1=p_2=p_3=k/3$

and $$w=\sum_{n=1}^{\infty} 3A_n .\ \sin \tfrac{2}{3}n\pi \tag{33}$$

where $$A_n=\frac{2s^2\theta_0 E\alpha f(h)}{nD\pi(1-\sigma)} \cdot \frac{1-e^{-\left(\frac{4n^2\pi^2}{3a^2}+c^2\right)\chi t}}{\left(\frac{4n^2\pi^2}{3a^2}+c^2\right)\left(\frac{4n^2\pi^2}{3a^2}+\beta_1^2\right)} . \tag{34}$$

In calculating the deflection one has to start from (29) and (34) with an assumed value of $\beta_1=0.01$ leading to a particular value of $\frac{2s^2\theta_0 E\alpha f(h)}{D\pi(1-\sigma)}$.

These values of β_1 and $\frac{2s^2\theta_0 E\alpha f(h)}{D\pi(1-\sigma)}$ determine the corresponding deflections from eqn. (33) for various values of χt as tabulated.

TABLE

χt	0	0.2	0.4	0.6	0.8	1.0
w	0	0.1100	0.1401	0.1598	0.1620	0.1708

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