STRUCTURE DEFINED BY A TENSOR FIELD $\phi(\neq 0)$ OF TYPE (1, 1) SATISFYING $\phi^p \pm \phi^{p-2} = 0$

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(f-structure and $\phi(4,\pm 2)$ structure have been studied by several authors (Yano 1969, 63 ; Yano *etal.* 1972). In this paper the authors have generalized these structures for any integer p, $(2 \leq p \leq n)$. The integrability conditions of this generalized structure manifold have been studied by defining three distributions Π_m, $\tilde{\Pi}_m$ and Π_{n-2m} in the differentiable manifold.

INTRODUCTION

Let M^n be an n-dimensional differentiable manifold of class C^∞ and let there be given a tensor field ϕ $(\neq 0)$ of type (1, 1) and of class c^∞ such that

$$\phi^p + \phi^{p-2} = 0 \qquad (2 \leq p < n) \tag{1.1}$$

where p is an integer.

Let the integer p be odd, $p = 2k+1$, $K = 1, 2, 3, \cdots\cdots$, etc. (Yano 1963, 1969).

Theorem 1.1—If we define the operators l and m as

$$l = (-1)^k \ \phi^{2k}, \quad m = I + (-1)^{k+1} \phi^{2k} \tag{1.2}$$

then these are complementary projection operators.

Proof: From (1.2) it is clear that

$$l + m = I.$$

Now

$$lm = (-1)^k \phi^{2k} - \phi^{4k}. \tag{1.3}$$

In consquence of (1. 1) we see that

$$\phi^{4k} = -\phi^{4k-2} = (-1)^2 \ \phi^{4k-4} = \ldots = \phi^{4k-2k}(-1)^{1k}.$$

From this and from (1.3) we have

$$lm = 0.$$

Similarly

$$ml = 0$$
$$l^2 = l$$
$$m^2 = m,$$

hence the result follows.

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Theorem 1.2—Let L and M be the dirtsibutions corresponding to the operators l and m respectively. The structure ϕ acts as an almost Complex structure on L while it acts on M as almost tangent structure of 2 $(k-1)$th order.

Proof: From (1.2) we see that

$$\phi^2 l = (-1)^k \phi^{2k+2} = (-1)^k(-\phi^{2k}) = -l. \tag{1.4}$$

Similarly

$$l\phi^2 = -l.$$

Thus ϕ acts on L as an almost complex structure. Now

and
$$\left.\begin{aligned} \phi^{2k-2}m &= \phi^{2k-2} - (-1)^k \phi^{4k-2} \\ \phi^{2k-1}m &= \phi^{2k-1} - (-1)^k \phi^{4k-1}. \end{aligned}\right\} \tag{1.5}$$

In consequence of (1.1) we get

$$\phi^{2k-1}m = 0 \tag{1.6}$$

hence the result follows.

Theorem 1.3—For odd values of p if we put

$$\bar{f}' = 2(-1)^k \phi^{2k} - I \tag{1.7}$$

then $\bar{f}'$ acts as an almost product structure.

Proof: In consequence of (1.1) and (1.7) we have

$$\bar{f}'^2 = 4\phi^{4k} - 4(-1)^k \phi^{2k} + I = I \tag{1.8}$$

hence the result follows.

Theorem 1.4—The structure ϕ given by (1.1) is not unique. Let μ be a non-singular vector valued linear function in V_n. Then ϕ' defined by

$$\mu(\phi'(X)) \underset{=}{\text{def}} \phi(\mu(X)) \tag{1.9}$$

also satisfies (1.1).

Proof : In consequence of (1.1) and (1.9) we get

$$\phi^{p-1}(\mu(\phi'(X))) = \phi^{p}(\mu(X)) = -\phi^{p-2}(\mu(X)) = -\mu(\phi'^{p-2}(X)). \tag{1.10a}$$

From (1.9) we get

$$\phi^{p-1}(\mu(\phi'(X))) = \mu(\phi'^{p}(X)). \tag{1.10b}$$

Adding (1.10*a*) and (1.10*b*) we get

$$\mu(\phi'^{p}(X) + \phi'^{p-2}(X)) = 0.$$

Since μ is non-singular

$$\phi'^{p} + \phi'^{p-2} = 0$$

which proves the result.

Note (1.1) : When p is odd let l'_1 and m'_1 be the operators for ϕ structure.

Theorem 1.5—We have

$$\mu(l'_1(X)) = l(\mu(X)) \tag{1.11a}$$

$$\mu(m'_1(X)) = m(\mu(X)). \tag{1.11b}$$

Proof : In consequence of (1.9) we get

$$\mu(l'_1(X)) = \mu\{(-1)^k \phi'^{2k}(X)\}) = (-1)^k \phi^{2k}(\mu(X)) = l(\mu(X))$$

which is (1.11*a*).

Again

$$\begin{aligned}\mu\{m'_1(X) &= \mu\{I+(-1)^{k+1}\phi'^{2k}\}(X)\\ &= \mu\{X+(-1)^{k+1}\phi'^{2k}(X)\}\\ &= \{I+(-1)^{k+1}\phi^{2k}\}(\mu(X))\\ &= m(\mu(X))\end{aligned}$$

which proves (1.11*b*).

Let in n-dimensional differentiable manifold with ϕ structure given by (1.1) we can always introduce a symmetric metric tensor g. Let g satisfy

$$g(\phi^{p-2}(X), \phi^{p-2}(Y)) = g(\phi^{p-1}), (X), Y) = g(X, \phi^{p-1}(Y)) \tag{1.12a}$$

$$g(\phi^{p-1}(X), \phi^{p-1}(Y)) = g(\phi^{p}(X), \phi(Y)). \tag{1.12b}$$

It is easy to see that repeated operators of barring X or Y yield the same set of equations and have no contradiction.

When p is odd, let us put

$$'m(X, Y) = g(m(X), Y) = g(X, m(Y))$$

where

$$m = I-(-1)^k \phi^{2k}.$$

We then have

$$g(\phi^{2k-1}(X), \phi^{2j-1}(Y)) = g(\phi^{2k}(X), Y) = g(X, \phi^{2k}(Y)) = (-1)^k g(X, Y) - (-1)^k\, 'm(X, Y). \tag{1.13}$$

Theorem 1.6—Let us put

$$g'(X, Y) = g(\mu(X), \mu(Y)) \tag{1.14}$$

then g' satisfies the equation of type (1.13), that is,

$$g'(\phi'^{2k-1}(X), \phi'^{2k-1}(Y)) = (-1)^k g'(X,Y) - (-1)^k\, 'm'(X,Y) \tag{1.15}$$

where

$$'m'(X, Y) = 'm(\mu(X), \mu(Y)).$$

Proof: In consequence of (1.14) and (1.9) we have

$$g'(\phi'^{2k-1}(X), \phi'^{2k-1}(Y)) = (\mu(\phi'^{2k-1}(X)), \mu(\phi'^{2k-1}(Y)))$$
$$= g(\phi^{2k-1}(\mu(X)), \phi^{2k-1}(\mu(Y)))$$
$$= (-1)^k g(\mu(X), \mu(Y)) - (-1)^k \times 'm(\mu(X), \mu(Y))$$
$$= (-1)^k g'(X,Y) - (-1)^k 'm'(X,Y)$$

which proves the result.

Theorem 1.7 – Let us put

$$\bar{f} \equiv \phi^{p'} \tag{1.16}$$

then $\bar{f}$ satisfies (1.1) if and only if p' is an odd integer.

Proof: In consequence of (1.1) and (1.16) we have

$$\bar{f}^{p} = \phi^{pp'}$$
$$= \phi^{p} \ldots \phi^{p} \ (p' \text{ times})$$
$$= -\phi^{p-2} \ldots -\phi^{p-2}$$
$$= (-1)^{p'} \phi^{(p-2)p'}$$

Let p' be an odd integer, then

$$\bar{f}^{p} = -\phi^{(p-2)p'} \tag{1.17}$$

From (1.16) we have

$$\bar{f}^{p-2} = \phi^{(p-2)p'}. \tag{1.18}$$

Adding (1.17) and (1.18) we get that f satisfies (1.1).

Conversely let us suppose that f satisfies (1.1) then

$$\bar{f}^{p} + \bar{f}^{p-2} = 0$$

which yields

$$\phi^{pp'} + \phi^{p'(p-2)} = 0 \tag{1.19}$$

or

$$(-1)^{p'} \phi^{p'(p-2)} + \phi^{p'(p-2)} = 0$$

This is true only when p′ is odd.

Let the integer p be even of the form $2k'$, $k' = 1, 2, 3, 4$. etc. (Yano *et al.* 1972).

Theorem 1.8—Let us define the operators l' and m' as

$$l' = (-1)^{k'+1} \phi^{2k'-2} m' = \mathrm{I} + (-1)^{k'+2} \phi^{2k'-2} \tag{1.20}$$

then these are complementary projection operators.

Proof: It follows the same pattern as of the proof of Theorem 1.1.

Theorem 1.9.—Let L' and M' be the distribution corresponding to the operators l' and m' respectively ϕ acts as an almost complex structure on L' and it acts on M' as an almost tangent structure of $(2k-3)$ th order.

Proof: It follows the same pattern as of the proof of the theorem (1.2).

Theorem 1.10—For even integer p, let us put

$$f' = 2(-1)^{k'+1}\phi^{2k'-2} - I \tag{1.21}$$

then f' acts as an almost product structure.

Proof: In consequence of (1.21) and (1.1) we get

$$f'^2 = 4\phi^{2(2k'-2)} - 4\phi^{2k'-2}(-1)^{k'+1} + I$$
$$= I$$

which proves the statement.

For even p let us put

$$'m'_1(X, Y) = g(m'(X), Y) = g(X, m'(Y)) \tag{1.22}$$

where

$$m' = I - (-1)^{k'+1}\,\phi^{2k'-2}$$

From (1.12*b*) we have

$$\begin{aligned} g(\phi^{2k'-1}(X),\ \phi^{2k'-1}(Y)) &= g(\phi^{2k'}(X),\ \phi(Y)) \\ &= g(-\phi^{2k'-2}(X), \phi(Y)) \\ &= (-1)^{k'+1}\,'m'_1(X, \phi(Y)) - (1)^{k'+1} \\ &\qquad \times g(X, \phi(Y)). \end{aligned} \tag{1.23}$$

Theorem 1.11—Let us put

$$g'(X,Y) = g(\mu(X), \mu(Y))$$

then g' satisfies the equation of type (1.23), that is

$$g'(\phi'^{2k'-1}(X), \phi'^{2k'-1}(Y))$$
$$= (-1)^{k'+1}\,''m_1'', (X, \phi'(Y)) - (-1)^{k'+1}\,g'(X, \phi'(Y))$$

where

$$''m_1''(X, \phi'(Y)) = 'm'_1\mu(X), \mu(\phi'(Y)).$$

Proof:

$$\begin{aligned} g(\phi'^{2k'-1}(X), \phi'^{2k'-1}(Y)) &= g(\mu(\phi'^{2k'-1}(X)), \mu(\phi'^{2k-1}(Y)) \\ &= g(\phi^{2k'-1}(\mu(X)), \phi^{2k'-1}(\mu(Y))) \\ &= (-1)^{k'+1}\,'m'_1(\mu(X)), \phi(\mu(Y)) \\ &\qquad -(-1)^{k'+1}g(\mu(X), \phi(\mu(Y))) \\ &= (-1)^{k'+1}\,'m'(\mu(X), \mu(\phi'(Y))) \\ &\qquad -(-1)^{k'+1}\,g(\mu(X), \mu(\phi'(Y))) \\ &= (-1)^{k'+1}\,''m''(X, \phi'(Y)) - (-1)^{k'+1}g'(X, \phi'(Y)) \end{aligned}$$

which proves the statement.

Theorem 1.12—Let l''_1 and m''_1 be the operators for ϕ' structure when p is even then

$$\mu(l_1''(X)) = l_1'(\mu(X))$$
$$\mu(m_1''(X)) = m_1'(\mu(X))$$

Proof: It follows the same pattern as of the proof of Theorem 1.5.

Note (1.2) : Hence forth we consider the values of $p > 2$.

Let in n-dimensional differentiable manifold with $\phi(p, p-2)$ structure there be a distribution Π_m of complex dimension m, a distribution $\tilde{\Pi}_m$ complex conjugate to m and distribution Π_{n-2m} have no direction in common and span togethear a liner manifold of dimension n, projections on Π_m, $\tilde{\Pi}_m$ and Π_{n-2m} being given by L, M, m respectively such that (Mishra *in press*)

$$\left.\begin{aligned} &(a)\quad 2L(X) = -\phi^{p-1}(X) - i\phi^{p-2}(X) \\ &(b)\quad 2M(X) = -\phi^{p-1}(X) + i\phi^{p-2}(X) \\ &(c)\quad m(X) = \phi^{p-1}(X) + \phi^{p-3}(X) \end{aligned}\right\} \tag{1.24}$$

Theorem 1.13—We have

$$L(M(X)) = L(m(X)) = M(L(X)) = M(m(X))$$
$$= m(L(X)) = m(M(X)) = 0.$$

Proof :

$$\begin{aligned} M(m(X)) &= \tfrac{1}{2}[-\phi^{p-1}(m(X)) + i\,\phi^{p-2}(m(X))] \\ &= -\phi^{p-1}[\phi^{p-1}(X) + \phi^{p-3}(X)]\tfrac{1}{2} + i\,\phi^{p-2}[\phi^{p-1}(X) + \phi^{p-3}(X)]\,.\tfrac{1}{2} \\ &= -\phi^{2p-2}(X).\tfrac{1}{2} - \phi^{2p-4}(X)\tfrac{1}{2} + i\,\phi^{2p-3}(X)\tfrac{1}{2} + i\,\phi^{2p-5}(X)\,.\,\tfrac{1}{2} \\ &= 0. \end{aligned}$$

Similarly others can also be proved.

Theorem 1.14—We have

$$\mu(L'_1(X)) = L(\mu(X))$$
$$\mu(M'_1(X)) = M(\mu(X))$$
$$\mu(m'_1(X)) = m(\mu(X))$$

where L_1, M_1, m_1 are the projections on Π_m, $\tilde{\Pi}_m$ and Π_{n-2m} for ϕ' structure.

Proof:
$$\begin{aligned} \mu(L'_1(X)) &= \tfrac{1}{2}\mu(-\phi'^{p-1}(X) - i\,\phi'^{p-2}(X)) \\ &= \tfrac{1}{2}(-\phi^{p-1}(\mu(X)) - i\,\phi^{p-2}(\mu(X))) \\ &= L(\mu(X)). \end{aligned}$$

Similarly others can also be done.

2. Integrability Conditions :

Ishihara and Yano (1964) obtained the integrability conditions for the distributions Π_r and Π_{n-r}. We will obtain here the integrability conditions for the distributions Π_m, $\tilde{\Pi}_m$ and Π_{n-2m} for this we need the following lemmas.

Lemma 2.1—We have

$$2dL(m(X), m(Y)) = \{\phi^{p-1}[\phi^{p-1}(X), \phi^{p-1}(Y)] + \phi^{p-1}[\phi^{p-1}(X), \phi^{p-3}(Y)] + \phi^{p-1}[\phi^{p-3}(X), \phi^{p-1}(Y)] + \phi^{p-1}[\phi^{p-3}(X), \phi^{p-3}(Y)]\} + i\{\phi^{p-2}[\phi^{p-1}(X), \phi^{p-1}(Y)) + \phi^{p-2}(\phi^{p-1}(X), \phi^{p-3}(Y)] + \phi^{p-2}(\phi^{p-3}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-3}(Y)]\} \quad (2.1)$$

$$2dM(m(X), m(Y)) = \{\phi^{p-1}[\phi^{p-1}(X), \phi^{p-1}(Y)] + \phi^{p-1}[\phi^{p-1}(X), \phi^{p-3}(Y)] + \phi^{p-1}[\phi^{p-3}(X)\ \phi^{p-1}(Y)] + \phi^{p-1}[\phi^{p-3}(X), \phi^{p-3}(Y)]\} - i\{\phi^{p-2}[\phi^{p-1}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-1}(X), \phi^{p-3}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-3}(Y)]\}. \quad (2.2)$$

Proof: In consequence of Theorem 1.13 we have

$$dL(m(X), m(Y)) = m(X)\, L(m(Y)) - m(Y)\,(L(m(X))) - L([m(X), m(Y)]) = -L([m(X), m(Y)])$$

$$2L(m(X), m(Y)]) = 2L([\phi^{p-1}(X) + \phi^{p-3}(X), \phi^{p-1}(X) + \phi^{p-3}(Y)]) = 2L([\phi^{p-1}(X), \phi^{p-1}(Y)]) + 2L([\phi^{p-1}(X), \phi^{p-3}(Y)]) + 2L([\phi^{p-3}(X), \phi^{p-1}(Y)]) + 2L([\phi^{p-3}(X), \phi^{p-3}(Y)]) \quad (2.3)$$

$$2L([m(X), m(Y)]) = -\phi^{p-1}[\phi^{p-1}(X), \phi^{p-1}(Y)] - \phi^{p-1}[\phi^{p-1}(X), \phi^{p-3}(Y)] - \phi^{p-1}[\phi^{p-3}(X), \phi^{p-1}(Y)] - \phi^{p-1}[\phi^{p-3}(X), \phi^{p-3}(Y)] - i\{\phi^{p-2}[\phi^{p-1}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-1}(X)\ \phi^{p-3}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-3}(Y)]\} \quad (2.4)$$

In consquence of (2.3) and (2.4), (2.1) follows. Similarly we can discuss for (2.2).

Lemma 2.2—We have

$$2dL(m(X), m(Y)) = \phi^{p-1}[m(X), m(Y)] + i\phi^{p-2}[m(X), m(Y)] \quad (2.5a)$$

$$2dM(m(X), m(Y)) = \phi^{p-1}[m(X)\ m(Y)] - i\phi^{p-2}[m(X), m(Y)] \quad (2.5b)$$

$$dm(M(X), M(Y)) = -\phi^{p-1}[M(X), M(Y)] - \phi^{p-3}[M(X), M(Y)] \quad (2.5c)$$

Proof: Using the definition (1.24*a*) in (2.3) we get (2.5*a*). Similarly we can prove (2.5*b*) and (2.5*c*) also.

Lemma 2.3 — We have

$$2dL(M(X), M(Y)) = \phi^{p-1}[M(X), M(Y)] + i\phi^{p-2}[M(X), M(Y)] \qquad (2.6a)$$

$$2dM(L(X), L(Y)) = \phi^{p-1}[L(X), L(Y)] - i\phi^{p-2}[L(X), L(Y)] \qquad (2.6b)$$

$$dm(L(X), L(Y)) = -\phi^{p-1}[L(X), L(Y)] - \phi^{p-3}[L(X), L(Y)] \qquad (2.6c)$$

Proof: It follows the same pattern as of the proof of Lemma 2.2.

Theorem 2.1—For the distribution Π_{n-2m} to the integrable it is necessary and sufficient that

$$\phi^{p-1}[\phi^{p-1}(X), \phi^{p-1}(Y)] + \phi^{p-1}[\phi^{p-1}(X), \phi^{p-3}(Y)] + \phi^{p-1}[\phi^{p-3}(X), \phi^{p-1}(Y)] + \phi^{p-1}[\phi^{p-3}(X), \phi^{p-3}(Y)] = 0 \qquad (2.7a)$$

equivalent to

$$\phi^{p-2}[\phi^{p-1}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-1}(X), \phi^{p-3}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-1}(Y)] + \phi^{p-2}[\phi^{p-3}(X), \phi^{p-3}(Y)] = 0 \qquad (2.7b)$$

or

$$\phi^{p-1}[m(X), m(Y)] = 0 \qquad (2.8a)$$

equivalent to

$$\phi^{p-2}[m(X), m(Y)] = 0. \qquad (2.8b)$$

Proof: The distribution Π_{n-2m} is given by

(*a*) $I(X)=0$, (*b*) $M(X)=0$, (*c*) $X=m(X)$.

In order that Π_{m-2m} be completely integrable it is necessary and sufficient that

$$L(X)=0, \quad M(X)=0$$

be completely integrable, that is,

$$dL(X, Y)=0, \quad dM(X, Y)=0$$

be satisfied by any vector satisfying (*c*). Hence

$$dL[m(X), m(Y)]=0, \; dM[m(X), m(Y)]=0.$$

Using lemmas 2.1 and 2.2 the result follows.

Theorem 2.2—In order that Π_m be completely integrable it is necessary and sufficient that

$$\phi^{p-1}[L(X), L(Y)] = 0 \qquad (2.9a)$$

equivalent to

$$\phi^{p-2}[L(X), L(Y)] = 0. \qquad (2.9b)$$

Proof: The distribution Π_m is given by

(*a*) $M(X)=0$, (*b*) $m(X)=0$, (*c*) $X=L(X)$.

In order that Π_m be completely integrable it is necessary and sufficient that $M=0$, $m=0$ be completely integrable, that is

$$dM(X, Y)=0, \quad dm(X, Y)=0$$

be satisfied by any vector satisfying (c) we get

$$dM(L(X), L(Y))=0, \quad dm(L(X), L(Y)]=0.$$

Using Lemma 2.3 we get the result.

Theorem 2.3.—In order that $\tilde{\Pi}_m$ be completely integrable it is necessary and sufficient that

$$\phi^{p-1}[M(X), M(Y)]=0 \qquad (2.10a)$$

equivalent to

$$\phi^{p-2}[M(X), M(Y)]=0 \qquad (2.10b)$$

Proof: It follows the same pattern as of the proof of Theorem 2.2.

We can write the results of Lemmas 2.1, 2.2 and 2.3 in terms of Nijenhuis tensor N as follows:

Remark (2.1): We have

$$8dL(M(X), M(Y))=\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-2}(Y)) + i\,\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-2}(Y)) \qquad (2.11a)$$

$$8dM(L(X), L(Y))=\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-2}(Y)) - i\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-2}(Y)) \qquad (2.11b)$$

$$\begin{aligned}2dL(m(X),m(Y))&=\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-2}(Y))-\phi^{p-1}N(\phi^{p-3}(X), \phi^{p-3}(Y))\\ &-\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-3}(Y))-\phi^{p-2}N(\phi^{p-3}(X), \phi^{p-2}(Y))\\ &+i\{\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-2}(Y))-\phi^{p-2}N(\phi^{p-3}(X), \phi^{p-3}(Y))\\ &+\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-3}(Y))+\phi^{p-1}N(\phi^{p-3}(X), \phi^{p-2}(Y))\}\end{aligned} \qquad (2.11c)$$

$$\begin{aligned}8dM(m(X), m(Y))&=\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-2}(Y))-\phi^{p-1}N(\phi^{p-3}(X), \phi^{p-3}(Y))\\ &-\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-3}(Y))-\phi^{p-2}N(\phi^{p-3}(X), \phi^{p-2}(Y))\\ &-i\{\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-2}(Y))-\phi^{p-2}N(\phi^{p-3}(X), \phi^{p-3}(Y))\\ &+\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-3}(Y))+\phi^{p-1}N(\phi^{p-3}(X), \phi^{p-2}(Y))\}\end{aligned} \qquad (2.11d)$$

In consequence of the above remark and the Theorem (2.1) we can state the following result.

Remark (2.2): The necessary and sufficient condition for the distribution Π_{n-2m} to be integrable is

$$\begin{aligned}&\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-2}(Y))-\phi^{p-1}N(\phi^{p-3}(X), \phi^{p-3}(Y))\\ &-\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-3}(Y))-\phi^{p-2}N(\phi^{p-3}(X), \phi^{p-2}(Y))=0\end{aligned} \qquad (2.12a)$$

equivalent to

$$\begin{aligned}&\phi^{p-2}N(\phi^{p-2}(X), \phi^{p-2}(Y))-\phi^{p-2}N(\phi^{p-3}(X), \phi^{p-3}(Y))\\ &+\phi^{p-1}N(\phi^{p-2}(X), \phi^{p-3}(Y))+\phi^{p-1}N(\phi^{p-3}(X), \phi^{p-2}(Y))=0.\end{aligned} \qquad (2.12b)$$

Remark 2.3: In order that Π_m and $\tilde{\Pi}_m$ be completely integrable it is necessary and sufficient that

$$\phi^{p-1} N(\phi^{p-2}(X), \phi^{p-2}(Y))=0$$

equivalent to

$$\phi^{p-2} N(\phi^{p-2}(X), \phi^{p-2}(Y))=0.$$

3. Structure Defined by a Tensor Field $\phi(\neq 0)$ Satisfying $\phi^p-\phi^{p-2}=0$

Let m^n be an n-dimensional differentiable manifold of class C^∞ and let there be given a tensor field $\phi(\neq 0)$ of type (1,1) and of class C^∞ such that

$$\phi^p-\phi^{p-2}=0 \; (2\leqslant p<n) \tag{3.1}$$

where p is an integer.

Let the integr p be odd, $p=2k+1$, K takes the values 1, 2, 3. .. etc.

Theorem 3.1—If we define

$$l_1=\phi^{2k}, \; m_1=I-\phi^{2k} \tag{3.2}$$

then these are complementary projection operators.

Proof: Here

$$l_1+m_1 = 0.$$

Also

$$l_1 m_1 = \phi^{2k}-\phi^{4k}.$$

In consequence of (3.1) we have

$$\phi^{4k} = \phi^{4k-2} = \phi^{4k-4} = \ldots = \phi^{2k}.$$

Hence

$$l_1 m_1 = 0.$$

Similarly

$$m_1 l_1 = 0$$

$$l_1^2 = l_1$$

$$m_1^2 = m_1.$$

Hence the result follows.

Theorem 3.2—The structure ϕ acts L_1 as almost product structure and it acts on M_1 as an almost tangent structure of $2(k-1)$-th order.

Proof: It follows the same pattern as of the proof of Theorem 1.2.

Theorem 3.3—Let p be even, $p = 2k'$, $k' = 1, 2, 3, 4, \ldots$ etc. If we define

$$l''_1 = \phi^{2k'-2}, \quad m''_1 = I-\phi^{2k'-2} \tag{3.3}$$

then these are complementary projection operators.

Proof: It follows the same pattern as of the proof of Theorem 3.1.

Theorem 3.4—ϕ acts on $L_1^{\cdot\cdot}$ as almost product structure and it acts on $M_1^{\cdot\cdot}$ as almost tangent structure of $(2k-3)$-th order where $L_1^{\cdot\cdot}$ and $M_1^{\cdot\cdot}$ are the distributions corresponding to the operators $l_1^{\cdot\cdot}$ and $m_1^{\cdot\cdot}$.

Proof : It follows the same pattern as of the proof of Theorem 3.2.

Theorem 3.5—The structure ϕ satisfying (3.1) is not unique. Let μ be a non-singular vector valued linear function in V_n. Then ϕ' defined by

$$\mu(\phi'(X)) \underset{=}{\text{def}} \phi(\mu(X)) \tag{3.4}$$

satisfies (3.1).

Proof: It follows the same pattern as of the proof of the Theorem 1.4.

When p is odd let l'_2 and m'_2 be the operators for ϕ'.

Theorem 3.6—We have for odd p

$$\mu(l'_2(X)) = l_1(\mu(X)) \tag{3.5a}$$

$$\mu(m'_2(X)) = m_1(\mu(X)). \tag{3.5b}$$

Proof: We have in consequence of (3.4)

$$\mu(l'_2(X)) = \mu(\phi'^{2k}(X)) = \phi^{2k}(\mu(X)) = l_1(\mu(X)).$$

Similarly (3.5*b*) also follows.

Let in n-dimensional differentiable manifold with $\phi(p, -(p-2))$ structure we can always introduce a symmetric metric tensor g satisfying

$$g(\phi^{p-2}(X), \phi^{p-2}(Y)) = g(\phi^{p-1}(X), Y) = g(X, \phi^{p-1}(Y)) \tag{3.6a}$$

$$g(\phi^{p-1}(X), \phi^{p-1}(Y)) = g(\phi^{p}(X), \phi(Y)). \tag{3.6b}$$

It is easy to see that repeated operators of barring X or Y yields the same set of equations and have no contradiction.

When p is odd, let us put

$$'m_1(X, Y) = g(m_1(X), Y) = g(X, m_1(Y)) \tag{3.7}$$

where

$$m_1 = I - \phi^{2k}.$$

We then have

$$g(\phi^{2k-1}(X), \phi^{2k-1}(Y)) = g(\phi^{k2}(X), Y) = g(X, \phi^{2k}(Y))$$
$$= g(X, Y) - 'm_1(X, Y) \tag{3.8}$$

Theorem 3.7—Let us put

$$g'(X, Y) = g(\mu(X), \mu(Y)) \tag{3.9}$$

then g' satisfies the equation of type (3.8) that is

$$g'(\phi'^{2k-1}(X), \phi'^{2k-1}(Y)) = g'(X, Y) - 'm'_1(X, Y) \tag{3.10}$$

where

$$'m'_1(X, Y) = 'm_1(\mu(X), \mu(Y)).$$

Proof: It follows the same pattern as of the proof of Theorem 1.6.

Theorem 3.8—For odd values of p, if we put

$$f_1 = 2\phi^{2k'} - I \tag{3.11}$$

then f_1 acts as an almost product structure.

Proof: In consequence of (3.1) and (3.11) we have

$$f_1^2 = 4\phi^{4k'} - 4\phi^{2k'} + I$$
$$= I$$

which proves the result.

Theorem 3.9—For even integer p, let us put

$$f_1' = 2\phi^{2k'-2} - I \tag{3.12}$$

then f'_1 acts as an almost product structure.

Proof: It follows the same pattern as of the proof of the Theorem 3.8.

Theorem 3.10—Let l_2'' and m_2'' be the operators for ϕ', when p is even, then

$$\left.\begin{aligned} \mu(l''_2(X)) &= l''_1(\mu(X)) \\ \mu(m''_2(X)) &= m''_1(\mu(X)). \end{aligned}\right\} \tag{3.13}$$

Proof: It follows the pattern of the proof of the Theorem (3.6).

Let us put

$$'m'_2(X, Y) = g(m''_1(X, Y) = g(X, m''_1(Y)) \tag{3.14}$$

where

$$m''_1 = I - \phi^{2k'-2}.$$

For $p = 2k'$ we have from (3.6*b*)

$$g(\phi^{2k'-1}(X), \phi^{2k'-1}(Y)) = g(\phi^{2k'}(X), \phi(Y)) = g(\phi^{2k'-2}(X), \phi(Y))$$
$$= g(X, \phi(Y)) - 'm'_2(X, \phi(Y)) \tag{3.15}$$

Theorem 3.11—Let us put

$$g'(X, Y) = g(\mu(X), \mu(Y))$$

then g' satisfies the equation of type (3.15) that is

$$g'(\phi'^{2k'-1}(X), \phi'^{2k'-1}(Y)) = g'(X, \phi'(Y)) = ''m_2''(X), \phi'(Y))$$

where

$$''m_2''(X, \phi'(Y)) = 'm_2'(\mu(X)), \mu(\phi'(Y)).$$

Proof: It follows the pattern of the proof of the Theorem 3.7.

Theorem 3.12—Let us put

$$\bar{f}_1 = \phi^{p'} \tag{3.16}$$

then $\bar{f}_1$ satisfies (3.1) for all values of p'.

Proof: In consequence of (3.1) and (3.14) we have

$$\begin{aligned} \bar{f}_1^p &= \phi^{pp'} \\ &= \phi^p \ldots \phi^p (p' \text{ times}) \\ &= \phi^{p-2} \ldots \phi^{p-2} (p' \text{ times}) \\ \bar{f}_1^p &= \phi^{p'(p-2)} \end{aligned}$$

Also

$$\bar{f}_1^{p-2} = \phi^{p'(p-2)}.$$

Hence

$$\bar{f}_1^p - \bar{f}_1^{p-2} = 0$$

which proves the result.

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