THE OPERATIONAL CALCULUS OF ASSOCIATED LEGENDRE TRANSFORMS—I

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This paper is concerned with the associated Legendre transform which is a new extension of the Legendre transform of Churchill. Several interesting operational properties of this new transform are proved in some detail. The associated Legendre transforms of some particular functions are obtained. It is shown that the operational properties of the Legendre transform follow as special cases of those of the associated Legendre transformation. As applications of this transform, the boundary value problems for the region interior to a sphere consisting of the modified Laplace equation in spherical polar coordinates with the Dirichlet or the Neumann boundary conditions on the sphere are solved. Some new results of the Legendre transform are also presented in this work.

1. INTRODUCTION

Churchill (1954, 1972) has first introduced the Legendre transform of a function and developed its operational calculus in some detail. He has proved several theorems concerning inverse Legendre transform, convolution property, operational properties of differential and inverse differential forms of interest. Several Legendre transforms of particular functions have been obtained. Finally, he has shown many applications of this transform by solving the Dirichlet and Neumann boundary value problems in potential theory which involve the spherical polar coordinates.

The object of this paper is to introduce the associated Legendre transform as a generalization of the Legendre transform of Churchill and to study its basic operational properties. Several theorems concerning the inverse transformation, operational properties of differential forms and the associated Lengendre transform of derivatives of a function are established. The associated Legendre transforms of some particular functions are derived. It is shown that the operational properties of the Legendre transform follow as special cases of those of the associated Legendre transformation. In order to show some applications of this new transform, the boundary value problems for the region interior to a sphere consisting of the modified Laplace equation in spherical polar coordinates with the Dirichlet or the Neumann boundary condition on the sphere are solved. Further, some new results of the Legendre transform are also presented.

2. Definition and Simple Properties of the Associated Legendre Transform

The associated Legendre transform of a function $F(x)$ is denoted by $f(n, m) = T_n^m \{F(x)\}$ and defined by the integral

$$f(n, m) = T_n^m \{F(x)\} = \int_{-1}^{1} (1-x^2)^{-\frac{m}{2}} P_n^m(x) F(x)\, dx \tag{2.1}$$

provided the integral exists in the sense of Lebesgue, where $P_n^m(x)$ is the associated Legendre function of the first kind and m, n are non-negative integers.

It follows from the definition that the associated Legendre transform is a linear integral transform of the discrete variables m and n. The linearity is evident from the result

$$T_n^m \{\alpha F(x)+\beta G(x)\} = \alpha T_n^m \{F(x)\}+\beta T_n^m \{G(x)\} \tag{2.2}$$

where α and β are arbitrary constants.

Substituting $x = \cos\theta$, (2.1) becomes

$$T_n^m \{ F(\cos\theta)\} = \int_0^{\pi} P_n^m (\cos\theta) (\sin\theta)^{1-m} F(\cos\theta)\, d\theta. \tag{2.3}$$

In particular, when $m = 0$, the above transformation reduces to the Legendre transform of Churchill (1954).

The associated Legendre transform has the following simple properties :

(*i*) $$T_n^m \{F(-x)\} = (-1)^n f(n, m). \tag{2.4}$$

In view of the substitution $x = -y$ and the property $P_n^m(-x) = (-1)^n P_n^m(x)$

$$T_n^m \{F(-x)\} = \int_{-1}^{1} (1-x^2)^{-\frac{m}{2}} P_n^m(x) F(-x)\, dx$$

$$= (-1)^n f(n, \mathrm{m})$$

(*ii*) $$T_n^m \{(1-x^2)^{\frac{m}{2}} P_r^m (x)\} = \frac{2\delta_{n r}}{(2n+1)} \frac{(n+m)\,!}{(n-m)\,!} . \tag{2.5}$$

If $F(x) = (1-x^2)^{\frac{m}{2}} P_r^m(x), \quad r \geq 0$

then

$$T_n^m\{(1-x^2)^{\frac{m}{2}} P_r^m(x)\} = \int_{-1}^{1} P_n^m(x)\, P_r^m(x)\, dx = \frac{2\delta_{nr}}{(2n+1)} \frac{(n+m)!}{(n-m)!}$$

where $\delta_{nr} = 0$ when $n \neq r$ and $\delta_{nr} = 1$, when $n = r$.

(*iii*) $$T_n^m\{(1-x^2)^{\frac{m}{2}}\, x\, P_{n-1}^m(x)\} = \frac{2}{(4n^2-1)} \frac{(n+m)!}{(n-m-1)!} \tag{2.6}$$

Using a result of Copson (1935, p. 300), this result can easily be verified.

(*iv*) $$T_n^m\{(1-x^2)^{\frac{m}{2}} x\, P_r^m(x)\} = 0 \text{ when } r \leqslant n. \tag{2.7}$$

In view of the result of Copson (1935, p. 300), (2.7) follows immediately.

(*v*) $$T_n^m\{(1-x^2)^{\frac{m}{2}-1} P_n^k(x)\} = \frac{\delta_{mk}}{m} \frac{(n+m)!}{(n-m)!}. \tag{2.8}$$

By virtue of the result (Copson, 1935, p. 299)

$$T_n^m\{(1-x^2)^{\frac{m}{2}-1} P_n^k(x)\} = \int_{-1}^{1} P_n^m(x)\, P_n^k(x)\, (1-x^2)^{-1} dx$$

$$= \frac{\delta_{mk}}{m} \frac{(n+m)!}{(n-m)!}.$$

(*vi*) $$T_n^m\{F(x)\} = \frac{(-1)^m}{2^n n!} \int_{-1}^{1} F^{(m+n)}(x)\, (1-x^2)^n\, dx. \tag{2.9}$$

Using the Rodrigues' formula for $P_n^m(x)$ and invoking repeated partial integration, we obtain

$$T_n^m\{F(x)\} = \frac{1}{2^n n!} \int_{-1}^{1} F(x) \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n\, dx.$$

$$= \frac{1}{2^n n!} \left\{ \left[F(x) \frac{d^{m+n-1}}{dx^{m+n-1}} (x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} F'(x) \frac{d^{m+n-1}}{dx^{m+n-1}} (x^2-1)^n dx \right\}$$

$$= \frac{(-1)}{2^n n!} \int_{-1}^{1} F'(x) \frac{d^{m+n-1}}{dx^{m+n-1}} (x^2-1)^n dx$$

$$= \dots \dots$$

$$= \frac{(-1)^m}{2^n n!} \int_{-1}^{1} F^{(m+n)}(x)\, (1-x^2)^n dx$$

In particular, when $m = 0$, this result gives

$$f(n) = T_n \{ F(x)\} = \frac{1}{2^n\, n!} \int_{-1}^{1} F^{(n)}(x)\, (1-x^2)^n\, dx \tag{2.10}$$

where $f(n) = T_n \{F(x)\}$ is the Legendre transform of Churchill (1954).

This result for the Legendre transform is believed to be new.

(*vii*)
$$T_n^m \{ F(x)\} = (-1)^m\, T_n \{F^{(m)}(x)\} \tag{2.11}$$

provided $F(x)$ vanishes at $x = \pm 1$.

We have

$$T_n^m \{ F(x)\} = \int_{-1}^{1} (1-x^2)^{-\frac{m}{2}}\, P_n^m(x)\, F(x)\, dx$$

$$= \int_{-1}^{1} F(x) \frac{d^m}{dx^m} P_n(x)\, dx$$

which yields (2.11) after repeated partial integration.

This result indicates a relationship between the Legendre and the associated Legendre transform.

3. The Inverse Associated Legendre Transform

The inverse transformation is given by

$$F(x) = (T_n^m)^{-1} \{ f(n, m)\} = \sum_{n=0}^{\infty} \frac{(2n+1)}{2} \frac{(n-m)!}{(n+m)!} f(n, m)\, (1-x^2)^{\frac{m}{2}}\, P_n^m(x) \tag{3.1}$$

where $-1 < x < 1$.

Proof: Since it is possible to expand an arbitrary function in series of the associated Legendre function, we have

$$F(x) = (1-x^2)^{\frac{m}{2}} \sum_{n=0}^{\infty} a_n\, P_n^m(x). \tag{3.2}$$

Multiplication of (3.2) by $(1-x^2)^{-\frac{m}{2}}\, P_r^m(x)$ and integration over $(-1, 1)$ give

$$\int_{-1}^{1} (1-x^2)^{-\frac{m}{2}}\, P_r^m(x)\, F(x)\, dx = \int_{-1}^{1} \left(\sum_{n=0}^{\infty} a_n\, P_r^m(x)\, P_n^m(x)\right) dx$$

$$= \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_r^m(x)\, P_n^m(x)\, dx$$

which is permissible due to uniform convergence of the series in $|x|<1$.

In view of the orthogonal property of the associated Legendre transform (Copson 1935), we obtain

$$a_n = \frac{(2n+1)}{2} \frac{(n-m)!}{(n+m)!} \int_{-1}^{1} (1-x^2)^{-\frac{m}{2}} P_n^m(x) \; F(x)\, dx$$

so that (3.2) gives the result (3.1.).

4. The Differential Operators and their Associated Legendre Transform

The differential operators denoted by $D[F(x)]$, $D^2[F(x)] \ldots D^k[F(x)]$ are defined by

$$D\,[F(x)] = \frac{d}{dx}\left[(1-x^2)\,\frac{d\,F(x)}{dx} + 2mx\,\frac{d\,F(x)}{dx}\right] \tag{4.1}$$

$$D^2\,[F(x)] = D[D\,[F(x)]\,] \tag{4.2}$$

$$\cdots$$

$$D^k\,[F(x)] = D[D^{k-1}\,[F(x)]\,] \tag{4.3}$$

where k is a positive integer.

Especially, when $x=\cos\theta$, the differential form (4.1) reduces to

$$D[F(\cos\theta)] = \frac{1}{\sin\theta}\,\frac{d}{d\theta}\,[\sin\theta\frac{d}{d\theta}\;F(\cos\theta)]$$

$$+2m\cot\theta\frac{d}{d\theta}\,F(\cos\theta). \tag{4.4}$$

Theorem 4.1—If $F'(x)$ is continuous and $F''(x)$ is bounded and integrable in each subinterval of $(-1,1)$; $T_n^m\{F(x)\}$ exists, and

$$(1-x^2)^{-\frac{m}{2}}\,F(x),\;(1-x^2)\,F'(x) = 0(1) \qquad \text{as} \qquad |\,x\,| \to 1. \tag{4.5a, b}$$

Then $T_n^m\{D\,[F(x)]\}$ exists and

$$T_n^m\{D\,[\,F(x)]\} = -(n+m)\,(n-m+1)\,f\,(n,m). \tag{4.6}$$

Proof: By partial integration, it follows that

$$T_n^m\,\{D\,[F(x)]\} = \left[\,(1-x^2)^{\,1-\frac{m}{2}}\,P_n^m(x)\,\frac{d\,F(x)}{dx}\right]_{-1}^{1}$$

$$+[\,2m\,x\,(1-x^2)^{-\frac{m}{2}}\,P_n^m\,(x)\,F(x)\,]_{-1}^{1}$$

$$-\int_{-1}^{1}(1-x^2)\,\frac{d\,F(x)}{dx}\left\{\frac{d}{dx}\left[\,(1-x^2)^{-\frac{m}{2}}\,P_n^m(x)\,\right]\right\}dx$$

$$-\,2m\int_{-1}^{1}F(x)\left\{\frac{d}{dx}\left[\,x\,(1-x^2)^{-\frac{m}{2}}\,P_n^m(x)\,\right]\right\}. \tag{4.7}$$

In view of (4.5*a*, *b*), the first two terms of (4.7) disappear. Another partial integration of the first integral and simplification of the second integral of (4.7) give

$$T_n^m \{D[F(x)]\} = -\left[(1-x^2)F(x)\left\{\frac{d}{dx}\left[(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right]\right\}\right]_{-1}^{1} +$$

$$+\int_{-1}^{1} F(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}\left[(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right]\right\} dx$$

$$-2m\int_{-1}^{1} F(x)\left\{x\frac{d}{dx}\left[(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right]+(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right\} dx$$

which by (4.5*a*) and the differential equation for the associated Legendre function (Copson, 1935) becomes

$$T_n^m \{D[F(x)]\} = -(n+m)(n-m+1) f(n, m).$$

Thus the theorem is proved and may be regarded as one of the most basic operational properties of the associated Legendre transform.

It is then possible to evaluate the associated Legendre transforms of higher order differential forms $D^2[F(x)]$, $D^3[F(x)]$ ...... $D^k[F(x)]$ by invoking conditions similar to those stipulated in Theorem 4.1.

In fact,

$$\begin{aligned} T_n^m \{D^2[F(x)]\} &= T_n^m \{D[D[F(x)]]\} \\ &= -(n+m)(n-m+1)\, T_n^m\{D[F(x)]\} \\ &= [(n+m)(n-m+1)]^2 f(n, m) \end{aligned} \tag{4.8}$$

provided the conditions similar to those of Theorem 4.1 are fulfilled.

For the differential form $D^3[F(x)]$, we obtain

$$T_n^m \{D^3[F(x)]\} = (-1)^3 [(n+m)(n-m+1)]^3 f(n, m). \tag{4.9}$$

In general, we thus obtain

$$T_n^m \{D^k[F(x)]\} = (-1)^k [(n+m)(n-m+1)]^k f(n,m) \tag{4.10}$$

provided conditions similar to Theorem 4.1 are satisfied and k is a positive integer.

Corollary 4.1 If $T_n^m \{D[F(x)]\} = -(n+m)(n-m+1) f(n, m)$

then

$$T_n^m \{\tfrac{1}{4} F(x) - D[F(x)]\} = [(n+\tfrac{1}{2})^2 - m(m-1)] f(n, m). \tag{4.11}$$

This result is obtained by replacing $(n+m)(n-m+1)$ in (4.6) by $(n+\frac{1}{2})^2 - m(m-1) - \frac{1}{4}$ and then rearranging the terms.

Further, if we substitute

$$(n+\tfrac{1}{2})^2-(m-\tfrac{1}{2})^2 \quad \text{for} \quad (n+m)\,)\,(n-m+1) \quad \text{in} \quad (4.6)$$

we obtain

$$T_n^m\{D^2[\,F(x)]\} = [(n+\tfrac{1}{2})^2-(m-\tfrac{1}{2})^2]^2 f(n,m) \tag{4.12}$$

which is, by binomial theorem,

$$= \sum_{r=0}^{2} (-1)^{r+2} \binom{2}{r} (n+\tfrac{1}{2})^{4-2r} (m-\tfrac{1}{2})^{2r} f(n,m). \tag{4.13}$$

More generally, we find

$$T_n^m\{D^k[F(x)\,]\} = \sum_{r=0}^{k} (-1)^{k+r} \binom{k}{r} (n+\tfrac{1}{2})^{2k-2r} (m-\tfrac{1}{2}) f(n,m) \tag{4.14}$$

Remark : Results (4.6)—(4.10) illustrate that the associated Legendre transform reduces a differential form into an algebraic form. Therefore, it would be possible to transform a differential equation into an algebraic equation which can possibly be solved easily. Then the solution of the given differential equation may be obtained by using the inverse transformation.

5. The Associated Legendre Transform of Derivatives

Theorem 5.1—If $F(x)$ is continuous in each subinterval of the interval $(-1, 1)$ and $G(x)$ is defined by

$$G(x) = \int_{-1}^{x} F(t)\,dt, \quad G(1)=0 \tag{5.1}$$

then

$$f(n,m) = T_n^m\{G'(x)\} = -\int_{-1}^{1} G(x) \frac{d^{m+1}}{dx^{m+1}} P_n(x)\,dx \tag{5.2}$$

and

$$T_n^m\{G(x)\} = f(0,m)-f(1,m),\ n=0 \tag{5.3a, b}$$

$$= \frac{f(n-1,m)-f(n+1,m)}{(2n+1)}, \quad n>0.$$

Proof: It follows from the definition of the associated Legendre transform combined with partial integration and

$$P_n^m(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n\, n\,!} \quad \frac{d^{m+n}}{dx^{m+n}} (1-x^2)^n \tag{5.4}$$

that

$$f(n,m) = T_n^m\{G'(x)\} = -\int_{-1}^{1} G(x) \frac{d^{m+1}}{dx^{m+1}} P_n(x)\,dx.$$

Then we obtain

$$f(n-1, m)-f(n+1, m)=\int_{-1}^{1} G(x)\frac{d^{m+1}}{dx^{m+1}} P_{n+1}(x)\, dx - \int_{-1}^{1} G(x)\frac{d^{m+1}}{dx^{m+1}} P_{n-1}(x)\, dx$$

which is, by

$$(2n+1)\ P_n(x)=P'_{n+1}(x)-P'_{n-1}(x) \tag{5.5}$$

$$=\int_{-1}^{1} G(x)\ \frac{d^m}{dx^m}\ \{(2n+1)\ P_n(x)\}\ dx$$

$$=(2n+1)\int_{-1}^{1} G(x)\ (1-x^2)^{-\frac{m}{2}} P_n^m(x)\ dx$$

$$=(2n+1)\ g\ (n, m).$$

This proves result (5. 3*b*).

Substituting $n = 0$ and $n = 1$ in (5.2) one can easily find (5.3*a*).

Corollary 5.1—If $G'(x)$ is a sectionally continuous function of x in $-1 < x < 1$, then

$$T_n^m\ \{ G'(x)\} = -\ [\ (2n-1)\ g(n-1, m)+(2n-5)\ g(n-3, m)+ ..\ +\ g(0, m)\ ],$$
$$n = 1. 3. 5. \dots \tag{5.6}$$

$$= -\ [\ (2n-1)\ g(n-1, m)+(2n-5)\ g(n-3, m)+ \dots +3g(1, m)\ ],$$
$$n = 2, 4, 6, \dots \tag{5.7}$$

$$= 0, \qquad n = 0. \tag{5.8}$$

These results can readily be verified by using (5.2) and (5.5).

6. The Associated Legendre Transforms of Particular Functions

(a)
$$T_n^m\ \{\ \log\ (1-x)\}= \frac{T_n^m \left\{ -1 - \frac{2mx}{1-x} \right\}}{(n+m)\ (n-m+1)}. \tag{6.1}$$

It follows from the differential form (4.1) that

$$D\ [\ \log\ (1-x)\ ] = -\left(1 + \frac{2mx}{1-x}\right) \tag{6.2}$$

so that

$$T_n^m\ \{D\ [\ \log\ (1-x)\ ]\ \} = \int_{-1}^{1} D\ [\ \log\ (1-x)\ ]\ P_n^m(x)\ (1-x^2)^{-\frac{m}{2}}\ dx$$

which is, by partial integration,

$$= \left[-(1+x)\,(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right]_{-1}^{1} + \int_{-1}^{1} (1+x)\frac{d}{dx}\left[(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right] dx$$

$$+2m \int_{-1}^{1} x\,(1-x^2)^{-\frac{m}{2}} P_n^m(x) \left[\frac{d}{dx} \log(1-x)\right] dx.$$

Since $(1+x) = -(1-x^2)\dfrac{d}{dx}\log(1-x)$, another partial integration yields

$$T_n^m \{D[\log(1-x)]\} = \left[-(1+x)\,(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right]_{-1}^{1}$$

$$-\left[(1-x^2)\log(1-x)\frac{d}{dx}\{(1-x^2)^{-\frac{m}{2}} P_n^m(x)\}\right]_{-1}^{1}$$

$$+\int_{-1}^{1} \log(1-x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\{(1-x^2)^{-\frac{m}{2}} P_n^m(x)\}\right] dx$$

$$+2m\int_{-1}^{1} x(1-x^2)^{-\frac{m}{2}} P_n^m(x)\frac{d}{dx}\log(1-x)\,dx.$$

Performing the final integration by parts, we obtain

$$T_n^m \{D[\log(1-x)]\} = \left[-(1+x)\,(1-x^2)^{-\frac{m}{2}} P_n^m(x)\right]_{-1}^{1}$$

$$+\int_{-1}^{1}\left\{\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\log(1-x)\right]+2mx\frac{d}{dx}\log(1-x)\right\}\times$$

$$\times(1-x^2)^{-\frac{m}{2}} P_n^m(x)dx. \tag{6.3}$$

By Theorem 4.1, (6.3) gives

$$T_n^m \{D[\log(1-x)]\} = -(n+m)\,(n-m+1)\,f(n,m)$$

so that

$$f(n,m) = -\frac{T_n^m\left\{-1-\dfrac{2mx}{1-x}\right\}}{(n+m)\,(m-n+1)},\quad f(n,m) = T_n^m\{\log(1-x)\}$$

which is the desired result.

(b) $$T_n^m \{(1-x^2)^m \frac{d^m}{dx^m}(1-2ax+a^2)^{-\frac{1}{2}}\} = \frac{2a^n}{(2n+1)}\,\frac{(n+m)\,!}{(n-m)\,!},\quad |a|<1. \tag{6.4}$$

Differentiating the generating function of the Legendre polynomial, it follows from the uniform convergence that

$$\frac{d^m}{dx^m}(1-2ax+a^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} a^n \frac{d^m}{dx^m} P_n(x). \tag{6.5}$$

Multiplying (6.5) by $(1-x^2)^{\frac{m}{2}} P_r^m(x)$, integrating over $(-1,1)$ and then using the orthogonal property of associated Legendre function, one gets result (6.4)

(c) $$T_n^m \left\{(1-x^2)^m \frac{d^m}{dx^m} (1-2ax+a^2)^{-\frac{3}{2}}\right\}=\frac{2a^n}{(1-a^2)} \frac{(n+m)!}{(n-m)!}. \qquad (6.6)$$

Multiplication of (6.4) by a and then differentiation gives

$$T_n^m\left\{(1-x^2)^m \frac{d^m}{dx^m}\left[(1-2ax+a^2)^{-\frac{3}{2}} (1-a^2)-(1-2ax+a^2)^{-\frac{1}{2}}\right]\right\}$$

$$=\frac{2n\,a^n}{(n+\frac{1}{2})} \frac{(n+m)!}{(n-m)!}$$

which reduces to (6.6) by virtue of (6.4).

(d) $$T_n^m \left\{\int_0^a (1-x^2)^m t^{\alpha-1} \frac{d^m}{dx^m}(1-2xt+t^2)^{-\frac{1}{2}}\, dt\right\}$$

$$=\frac{2\,a^{n+\alpha}}{(2n+1)\,(n+\alpha)} \frac{(n+m)!}{(n-m)!},\ |a|<1. \qquad (6.7.)$$

From (6.4), we have

$$T_n^m\left\{(1-x^2)^m \frac{d^m}{dx^m} (1-2xt+t^2)^{-\frac{1}{2}}\right\}=\frac{2t^n}{(2n+1)} \frac{(n+m)!}{(n-m)!}$$

so that after multiplication by $t^{\alpha-1}$, $\alpha > 0$ and integration with respect to t over $(0, a)$ we obtain

$$T_n^m\left\{ \int_0^a (1-x^2)^m t^{\alpha-1} \frac{d^m}{dx^m} (1-2xt+t^2)^{-\frac{1}{2}}\, dt \right\}$$

$$=\int_0^a \frac{t^{n+\alpha-1}}{(n+\frac{1}{2})} \frac{(n+m)!}{(n-m)!}\, dt = \frac{2\,a^{n+\alpha}}{(2n+1)\,(n-\alpha)} \frac{(n+m)!}{(n-m)!}.$$

(e) $$T_n^m \left\{(1-x^2)^m \frac{d^m}{dx^m}\left[\log\left\{ \frac{a-x+(1-2ax+a^2)^{\frac{1}{2}}}{1-x}\right\}\right]\right\}$$

$$=\frac{2\,a^{n+1}}{(2n+1)\,(n+1)} \frac{(n+m)!}{(n-m)!},\ |a| < 1. \qquad (6.8)$$

Substituting for $(1-2ax+a^2)^{-\frac{1}{2}}$ in (6.4) and integrating with respect to a, the above result follows immediately.

(f) $$T_n^m \left\{ (1-x^2)^m \frac{d^m}{dx^m}\left[2a(1-2ax+a^2)^{-\frac{1}{2}} -\log \frac{a-x+(1-2ax+a^2)^{\frac{1}{2}}}{1-x}\right]\right\}$$

$$= \frac{2a^{n+1}}{(n+1)} \frac{(n+m)!}{(n-m)!},\ |a| < 1. \qquad (6.9)$$

This result is readily verified by subtracting (6.8) from (6.6).

(g) $$T_n^m \left\{ (1-x^2)^m \frac{d^m}{dx^m} \log \frac{1-ax+(1-2ax+a^2)^{\frac{1}{2}}}{2} \right\}$$

$$= \frac{2a^n}{n(2n+1)} \frac{(n+m)!}{(n-m)!}, \quad n > 0, m \geq 0$$

$$=0, n=0, m \geq 0 \tag{6.10}$$

where $|a| < 1$.

Upon multiplying (6.4) by $\frac{1}{a}$ we can write the result as

$$T_n^m \left\{(1-x^2)^m \frac{d^m}{dx^m} \left[a^{-1}(1-2ax+a^2)^{-\frac{1}{2}} -1\right]\right\}=0, n=0, m= \geq 0 \tag{6.11a}$$

$$= \frac{2a^{n-1}}{(2n+1)} \frac{(n+m)!}{(n-m)!}, \quad n > 0, m \geq 0. \tag{6.11b}$$

In view of the result

$$a^{-1} [(1-2ax+a^2)^{-\frac{1}{2}}-1]= \frac{1}{a}\left[\frac{a(x^2-1)(1-2ax+a^2)^{-\frac{1}{2}}-a(x^2-1)}{a(x^2-1)}\right]$$

$$= \frac{1}{a}\left[\frac{-x(1-2ax+a^2)^{\frac{1}{2}}+(ax-1)(a-x)(1-2ax+a^2)^{-\frac{1}{2}}+x(1-ax)+(a-x)}{a(x^2-1)}\right]$$

$$=-\left[\frac{\{(a-x)(1-2ax+a^2)^{-\frac{1}{2}}-x\}\{(1-ax)-(1-2ax+a^2)^{\frac{1}{2}}\}}{(1-ax)^2-(1-2ax+a^2)}\right]$$

$$=-\left\{\frac{(a-x)(1-2ax+a^2)^{-\frac{1}{2}}-x}{(1-ax)+(1-2ax+a^2)^{1/2}}\right\}$$

we derive

$$T_n^m\{(1-x^2)^m \frac{d^m}{dx^m}\left[\frac{x-(a-x)(1-2ax+a^2)^{-\frac{1}{2}}}{(1-ax)+(1-2ax+a^2)^{1/2}}\right]$$

$$= \frac{2a^{n-1}}{(2n+1)}\frac{(n+m)!}{(n-m)!}, \quad n > 0, m \geq 0$$

$$= 0, \quad n = 0, \quad m \geq 0.$$

Integration with respect to a in $(0, a)$ yields result (6.10)

(h) $$T_n^m \left\{(1-x^2)^m \frac{d^m}{dx^m}\left[(1-2ax+a^2)^{-\frac{1}{2}} - \frac{1}{2}\log \frac{1-ax+(1-2ax+a^2)^{\frac{1}{2}}}{2}\right]\right\}$$

$$= \frac{(n+m)!}{(n-m)!} \frac{a^n}{n}, \quad |a| < 1. \tag{6.12}$$

A simple combination of (6.4) and (6.10) gives the above result.

(i) $T_n^m\{F(x)\} = 0, \qquad n = 0, \qquad m > 0$

$$= 1, \qquad n = 0, \qquad m = 0 \tag{6.13}$$

$$= \frac{1}{(2n+1)}\left[\frac{d^m}{dx^m}\left\{P_{n+1}(x) - P_{n-1}(x)\right\}\right]_0^1, \; n > 0, \; m \geq 0$$

where $F(x)=1$, $0 < x < 1$ and $F(x) = 0$, $-1<x<0$.

It follows frcm the definition and (5.5) that

$$T_n^m\{1\} = \frac{1}{2n+1}\int_0^1 \frac{d^m}{dx^m}\{P'_{n+1}(x) - P'_{n-1}(x)\}\,dx$$

$$= \frac{1}{2n+1}\left[\frac{d^m}{dx^m}\{P_{n+1}(x) - P_{n-1}(x)\}\right]_0^1 \; n > 0, \; m \geq 0.$$

Further

$$T_n^m\{1\} = \int_0^1 \frac{d^m}{dx^m} P_0(x)\,dx = 0, \quad \text{when } n=0, \; m > 0$$

and

$$T_0^0\{1\} = 1.$$

(j) $$T_n\{e^{i\alpha x}\} = \left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}} 2^n J_{n+\frac{1}{2}}(\alpha) \tag{6.14}$$

where $T_n\{F(x)\} = f(x)$ denotes the Legendre transform of $F(x)$ defined by Churchill (1954) as

$$f(n) = T_n\{F(x)\} = \int_{-1}^1 F(x)\,P_n(x)\,dx \tag{6.15}$$

which is also obtained from (2.1) when $m = 0$.

The above result follows directly from (6.15) combined with the result of Copson (1935, p. 341) involving the Bessel function $J_\nu(x)$ of the first kind of order ν.

(k) $$T_n\{(1-x^2)^{-\frac{1}{2}}\} = \pi P_n^2(0). \tag{6.16}$$

With the aid of the result of Copson (1935, p. 310), one gets this result.

(l) $$T_n\{\tfrac{1}{2}(t-x)\} = Q_n(t) \tag{6.17}$$

where $Q_n(t)$ is the Legendre function of the second kind.

Result (6.17) is an immediate consequence of the integral formula for $Q_n(t)$

Remark : Results (6.14)—(6.17) are believed to be new.

7. Applications to Boundary Value Problems

We consider the boundary value problem for the region interior to the unit sphere $r=1$ involving the modified Laplace equation in spherical polar coordinates as

$$\nabla^2 U = \frac{2m}{r^2}\left[\cot\theta \frac{\partial}{\partial\theta} + (m+n)\right] U \tag{7.1}$$

and the function $U(r, \theta, \phi)$ or its normal derivative is prescribed on the boundary of the sphere $r=1$.

To solve these boundary value problems, we assume that U is independent of the spherical coordinate ϕ, U and its partial derivatives of the first and the second orders are continuous inside the sphere. Substituting $x = \cos\theta$, eqn. (7.1) reduces to the form

$$r\frac{\partial^2}{\partial r^2}(r\,U) + \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial U}{\partial x}\right] + 2m\left[x\frac{\partial U}{\partial x} - (m+n)\,U\right] = 0,$$

$$r < 1, \quad -1 < x < 1. \tag{7.2}$$

This equation has to be solved subject to the Dirichlet boundary condition

$$U(r, x) = F(x) \quad \text{on } r=1-0, \quad -1 < x < 1 \tag{7.3}$$

or Neumann boundary condition

$$\frac{\partial U}{\partial r} + G(x) \quad \text{on } r=1-0. \quad -1,<x<1. \tag{7.4}$$

We apply the associated Legendre transformation $u(r, m, n)$ of $U(r, x)$ defined by (2.1) to solve the above boundary value problems. In view of this transformation and its basic operational property (4.6), the above problems reduce to the problems

$$r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} - (m+n)(m-n+1)\,u=0, \; r<1, \tag{7.5}$$

$$u(r, n, m,) = f(n, m) \text{ on } r=1-0, \tag{7.6}$$

or

$$\frac{du}{dr} = g(n, m), \; r=1-0 \tag{7.7}$$

where m and n are non-negative integers.

The solution of (7. 5) with (7.6) is

$$u(r, n, m) = f(n, m)\, r^{n+m}. \tag{7.8}$$

Thus the inverse transformation gives the formal solution

$$U(r, x) = \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(n-m)!}{(n+m)!}(1-x^2)^{\frac{m}{2}}(2n+1)\, f(n, m)\, P_n^m(x) r^{n+m}, \; (r\leq 1, \; -1<x<1) \tag{7.9}$$

The solution of (7.5) with (7.7) has the form

$$u(r, n, m) = (g\, n, m)\,\frac{r^{n+m}}{n+m} \tag{7.10a}$$

$$= \text{constant} = A \text{ (say)}, \quad n=m=0 \tag{7.10b}$$

so that the inverse transformation gives the solution

$$U(r,x)=A+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n+1)\,(n-m)!}{(n+m)\,(n+m)\,!}(1-x^2)^{\frac{m}{2}}\,g\,(n,m)\,r^{n+m}P_n^m\,(x). \tag{7.11}$$

Similarly, the solution of (7.2) can readily be obtained subject to the mixed boundary condition

$$\frac{\partial U}{\partial r}+\alpha U=F\,(x),\text{ on } r=1-0,\ -1<x<1 \tag{7.12}$$

where α is a positive constant.

The solution of (7.5) subject to the transformed boundary condition (7.12) has the form

$$u\,(r,\,n,\,m)=\frac{f\,(n,m)}{(n+m+\alpha)}\,r^{n+m},\ 0\leq r\leq 1 \tag{7.13}$$

so that the inversion yields the solution

$$U\,(r,\,x)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(n-m)\,!\,(2n+1)}{(n+m)!\ (n+m+\alpha)}\,(1-x^2)^{\frac{m}{2}}f\,(n,m)P_n^m(x)\,r^{n+m},$$

$$(0\leqslant r\leq 1,\ -1<x<1).$$

8. Concluding Remarks

The above analysis reveals that the properties of the associated Legendre transform reduce to those of the Legendre transform of Churchill when $m=0$.

Although several operational properties of the associated Legendre transform including applications to boundary value problems have been presented in this paper, the convolution property as well as applications to boundary value problems of physical interest is under way. This work will be reported in a subsequent paper.

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