ON H-CURVATURE TENSORS IN ALMOST PRODUCT AND ALMOST DECOMPOSABLE MANIFOLD

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In this paper we have obtained the expressions for H-projective, H-conformal, H-concircular and H-conharmonic curvature tensors in almost product and almost decomposable manifold and have studied their properties in the light of pure and hybrid-property of a tensor in an almost product and almost decomposable manifold.

1. INTRODUCTION

A$_n$ n-dimensional manifold M_n of differentiability class C^{r+1} endowed with a real vector valued linear function f, a positive definite Riemannian metric g together with a Riemannian connexion D satisfying

$$\bar{\bar{X}} = X, \qquad \bar{X} \overset{def}{=} f(X) \quad \ldots \quad (1{\cdot}1)$$

$$g(\bar{X},\bar{Y}) = g(X,Y) \quad \ldots \quad (1{\cdot}2)$$

$$(D_X f)(Y) = 0 \quad \ldots \quad (1{\cdot}3)$$

for arbitrary vector fields X,Y,Z..........is called an almost product and almost decomposable manifold. Thus from (1·3) we see that an almost product and an almost decomposable manifold must have a symmetric f-connexion.

A linear function A is said to be pure in the two slots in an almost product and almost decomposable manifold if (Yano 1965)

$$A(\bar{X},\bar{Y}) - A(X,Y) = 0 \quad \ldots \quad (1{\cdot}4a)$$

and is said to be hybrid (Yano 1965) if

$$A(\bar{X},\bar{Y}) + A(X,Y) = 0. \quad \ldots \quad (1{\cdot}4b)$$

Let R, $'R$ and Ric be curvature, associated curvature and Ricci curvature tensors respectively then $'R$ is pure in the first (last) two slots and Ric is pure in its two slots (Mishra 1970).

2. H-CURVATURE TENSORS

Sinha (1976) has defined H-projective curvature tensor P, in almost product manifold, of a symmetric f-connexion by

$$P(X,Y,Z) = R(X,Y,Z) + Q(Y,Z)X - Q(X,Z)Y - \{Q(X,Y) - Q(Y,X)\} Z + Q(Y,\bar{Z})\bar{X} - Q(X,\bar{Z})\bar{Y} - \{Q(X,\bar{Y}) - Q(Y,\bar{X})\}\bar{Z}, \quad .. \quad (2\cdot1)$$

where

$$Q(X,Y) = -\{\text{Ric}(X,Y) + 1/n - 2(O(\text{Ric}(X,Y) + \text{Ric}(Y,X))\}/(n+2). \quad (2\cdot2)$$

Therefore H-projective curvature tensor in an almost product and almost decomposable manifold of a Riemannian connexion is given by (2·1). We know that a Ricci tensor is a symmetric tensor and

$$O\ \text{Ric}(X,Y) = \text{Ric}(\bar{X},\bar{Y}) + \text{Ric}(X,Y) = 2\ \text{Ric}(X,Y)$$

in an almost product and almost decomposable manifold. Consequently (2·2) gives

$$Q(X,Y) = -\text{Ric}(X,Y)/(n-2).$$

Hence H-projective curvature tensor in an almost product and almost decomposable manifold is given by

$$P(X,Y,Z) = R(X,Y,Z) + 1/n-2\ \{\text{Ric}(X,Z)Y - \text{Ric}(Y,Z)X + \text{Ric}(\bar{X},Z)\bar{Y} - \text{Ric}(\bar{Y},Z)\bar{X}\}. \quad .. \quad .. \quad .. \quad (2\cdot3)$$

Let us define trilinear vector valued function Q, T and S by

$$Q(X,Y,Z) = R(X,Y,Z) + 1/n-4\{\text{Ric}(X,Z)Y - \text{Ric}(Y,Z)X + \text{Ric}(\bar{X},Z)\bar{Y} - \text{Ric}(\bar{Y},Z)\bar{X} + r(Y)g(X,Z) - r(X)g(Y,Z) + r(\bar{Y})g(\bar{X},Z) - r(\bar{X})g(\bar{Y},Z)\} - K\{g(X,Z)Y - g(Y,Z)X + g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X}\}/(n-2)\ (n-4), \quad (2\cdot4)$$

$$T(X,Y,Z) = R(X,Y,Z) + K\{g(X,Z)Y - g(Y,Z)X + g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X}\}/n(n-2), \quad (2.5)$$

$$S(X,Y,Z) = R(X,Y,Z) + \{\text{Ric}(X,Z)Y - \text{Ric}(Y,Z)X + \text{Ric}(\bar{X},Z)\bar{Y} - \text{Ric}(\bar{Y},Z)\bar{X} + r(Y)g(X,Z) - r(X)g(Y,Z) + r(\bar{Y})g(\bar{X},Z) - r(\bar{X})g(\bar{Y},Z)\}/(n-4) \quad (2\cdot6)$$

where $\text{Ric}(X,Y) = g(r(X)Y)$ and $K = C_1^1 r)$. These are constructed from conformal, concircular and conharmonic curvature tensors by taking account of the resemblance between the projective curvature and H-projective curvature tensors. We call these tensors as H-conformal, H-concircular and H-conharmonic curvature tensors.

Let us now define tensors of type (1,3) by (Sinha 1973).

$$A(X,Y,Z) = \text{Ric}(X,Z)Y - \text{Ric}(Y,Z)X, \quad .. \quad .. \quad .. \quad (2\cdot7a)$$

$$B(X,Y,Z) = g(X,Z)Y - g(Y,Z)X, \quad .. \quad .. \quad .. \quad (2\cdot7b)$$

$$C(X,Y,Z) = g(X,Z)r(Y) - g(Y,Z)r(X). \quad .. \quad .. \quad .. \quad (2\cdot7c)$$

The expressions for different H-curvature tensors in terms of the tensors A,B and C are

$$P(X,Y,Z) = R(X,Y,Z) + \{A(\bar{X},\bar{Y},Z) + A(X,Y,Z)\}/n-2 \quad \ldots \quad (2\cdot8a)$$

$$Q(X,Y,Z) = R(X,Y,Z) + \{A(\bar{X},\bar{Y},Z) + A(X,Y,Z) + C(\bar{X},\bar{Y},Z) + C(X,Y,Z) \\ /(n-4) - K\{B(\bar{X},\bar{Y},Z) + B(X,Y,Z)\}/(n-2)\,(n-4) \ldots \quad (2\cdot8b)$$

$$T(X,Y,Z) = R(X,Y,Z) + K\{B(\bar{X},\bar{Y},Z) + B(X,Y,Z)\}/n(n-2) \quad \ldots \quad (2\cdot8c)$$

$$S(X,Y,Z) = R(X,Y,Z) + \{A(\bar{X},\bar{Y},Z) + A(X,Y,Z) + C(\bar{X},\bar{Y},Z) + C(X,Y,Z)\} \\ /(n-4) \quad \ldots \quad \ldots \quad \ldots \quad (2\cdot8d)$$

Theorem 2.1—In order that all the tensors A,B, and C coincide it is necessary and sufficient that the manifold M_n be an Einstein manifold of constant sectional curvature equal to n.

PROOF : Let A coincide with B. Then contracting Y in (2·7a) and (2·7b) and equating the result, we get $\text{Ric}(X,Z) = g(X,Z)$ which proves the statement. Similarly if A concides with C or B concides with C, we have the same.

Conversely, let M_n be an Einstein manifold of constant scalar curvature n. Then $\text{Ric}(X,Y) = g(X,Y)$ and $r(X) = X$ which on substitution in (2·7) gives the result.

Theorem 2.2—The tensors A, B and C in almost product and almost decomposable manifold M_n of constant sectional curvature are either pure or hybrid in all the slots according as $\bar{X} = X$ or $\bar{X} = -X$ for arbitrary vector field X in M_n.

PROOF : Mishra (1970) has proved that an almost product and an almost decomposable manifold can not be of constant curvature unless $\bar{X} = \pm X$. Therefore if M_n is of constant scalar curvature then either $\bar{X} = X$ or $\bar{X} = -X$. Barring X, Y and Z in (2·7) and using (1·2) and the fact that Ric is pure in the two slots, we have

$$A(\bar{X},\bar{Y},Z) = \text{Ric}(X,Z)\bar{Y} - \text{Ric}\,(Y,Z)\bar{X}$$

$$B(X,Y,Z) = g(X,Z)\bar{Y} - g(Y,Z)\bar{X}$$

$$C(X,Y,Z) = g(X,Z)r(\bar{Y}) - g(\,Y,Z)r(\bar{X})$$

which show that A,B,C are pure or hybrid in all the slots according as $\bar{X} = X$ or $\bar{X} = -X$.

Following Theorems can be proved easily.

Theorem 2.3—In order that H-projecture (H-concircular) curvature tensor in an almost product and almost decomposable manifold be curvature tensor it is necessary and sufficient that $A(X,Y,Z)$ be hybrid ($K=0$ or $B(X,Y,Z)$ be hybrid) in the first two slots.

Theorem 2.4—A necessary and sufficient condition for H-conformal curvature tensor in an almost product and almost decomposable manifold coincides with

H-conharmonic curvature tensor is that either $K=0$ or $B(X,Y,Z)$ is hybrid in the first two slots.

From Theorem 2.3 and Theorem 2.4, we have

"A necessary and sufficient condition that H-conformal curvature tensor coincides with H-conharmonic curvature tensor is that the H-concircular curvature tensor equals to the Riemannian curvature tensor".

Theorem 2.5—In an almost product and almost decomposable manifold of constant sectional curvature all the three tensors A, B, C are pure in any two slots.

Theorem 2.6—In an almost product and almost decomposable manifold H-conformal, H-concircular, H-conharmonic and curvature tensors are linearly related as

$$(n-4)\ \{Q(X,Y,Z)-S(X,Y,Z)\}\ +n\ \{T(X,Y,Z)-R(X,Y,Z)\}\ =0.$$

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