ABSOLUTE BOREL SUMMABILITY OF A FOURIER SERIES AND ITS CONJUGATE SERIES

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Recently Chandra and Kathal (1973) obtained the sufficient conditions to ensure the absolute Borel Summability of Fourier series, its conjugate series and their derived series. In this paper, we ensure the absolute Borel sumability of Fourier series and its conjugate series, at a point $t=x$, under the different set of conditions imposed upon the generating functions of the Fourier and its conjugate series.

1. DEFINITIONS AND NOTATIONS

Let $\sum_{n=1}^{\infty} a_n$ be a given infinite series. Then $\sum_{n=1}^{\infty} a_n$ is absolutely summable by Borel's exponential method or, symbolically, $\sum_{n=1}^{\infty} a_n \in | B |$ if,

$$\int_0^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} a_{n+1} \right| dy < \infty. \tag{1.1}$$

(See Borwein and Shawyer 1966, Chandra and Kathal 1973).

Let $f \in L\ (-\pi, \pi)$ and be 2π—periodic. Without any loss of generality the constant term of the Fourier series of f can be taken to be zero so that its Fourier series and conjugate series of the Fouricr series be the following respectively :

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n (t) \tag{1.2}$$

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n (t). \tag{1.3}$$

We use the following notations throughout the paper : Let r be a non-negative integer.

$$\phi\ (t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\} \tag{1.4}$$

$$\psi\ (t)=\tfrac{1}{2}\{f\ (x+t)-f\ (x-)\ t\} \tag{1.5}$$

$$\phi_a(t)=\frac{\alpha}{t^a}\int_0^t (t-u)^{a-1}\phi\ (u)\ du\quad (\alpha>0) \tag{1.6}$$

and similarly we define $\psi_a(t)$. $(\alpha>0)$. For a function F, we also write

$$(F\ (nt))_r=\left[\left(\frac{\partial}{\partial y}\right)^r(F\ (ny))\right]_{y=t} \tag{1.7}$$

and

$$(F\ (nt))_{-r}=[\int\int\int\ldots F\ (ny)\ (dy)^r]_{y=t} \tag{1.8}$$

whenever the right-hand sides exist.

2. Introduction

In a recent paper, Chandra and Kathal (1973) gave the results concerning $|B|$ summability of the Fourier series, its conjugate series and their derived seriers. In the present paper we prove the following theorems .

Theorem 1—Let α be a positive integer. Then $t^{-a}\ \phi_a\ (t)\in BV\ (0,\pi)$implies that $\sum_{n=1}^{\infty} A_n(x)\in |B|$.

Theorem 2—LeI α be a positive integer. Then $t^{-a}\psi_a(t)\in BV\ (0,\pi)$ implies that $\sum_{n=1}^{\infty} B_n\ (x)\in |B|$.

3. Lemmas

We shall use the following lemmas in the proof of the theorems :

Lemma 1—Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Then $\sum_{n=1}^{\infty} a_n\in |B|$.

The proof, being straight forward, has been omitted.

Lemma 2—For any non-negative integer s

$$\sum_{n=s}^{\infty}\frac{y^n}{n\,!}(n+1)^s\ \exp\ (i\ (n+1)\ t)\sum_{j=0}^{\infty} C_j\sum_{n=s}^{\infty}\frac{y^n}{(n-j)\,!}\exp\ (\ i\ (n+1)\ t\ ),$$

where C_j are positive constants, depending upon s and j, and not necessarily the same at each occurrence.

Proof: $\sum_{n=s}^{\infty} \frac{y^n}{n!}(n+1)^s \exp(i(n+1)t)$

$$= \sum_{n=s}^{\infty} \frac{y^n}{n!} n(n-1)\ldots(n-s+1) \frac{(n+1)^s}{n(n-1)\ldots(n-s+1)} \exp(i(n+1)t)$$

$$= \sum_{n=s}^{\infty} \frac{y^n}{(n-s)!}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n-1}\right)\ldots\left(1+\frac{s}{n+1-s}\right) \exp(i(n+1)t)$$

$$= \sum_{n=s}^{\infty} \frac{y^n}{(n-s)!}\left[1+\left(\frac{1}{n}+\frac{2}{n-1}+\ldots+\frac{s}{n+1-s}\right)+\ldots + \frac{s!}{n(n-1)\ldots(n-s+1)}\right] \exp(i(n+1)t)$$

$$= \sum_{n=s}^{\infty} \frac{y^n}{(n-s)!} \exp(i(n+1)t) + \sum_{n=s}^{\infty} \frac{y^n}{(n-s)!\,n} \exp(i(n+1)t)$$

$$+\ldots+\sum_{n=s}^{\infty} \frac{y^n}{(n-s)!} \frac{s!}{n(n-1)\ldots(n-s+1)} \exp(i(n+1)t)$$

$$= \sum_{i=1}^{2^s} \Sigma_i, \text{ say.}$$

Now, we shall illustrate that the above 2^s-summations are the constant multiples of $(s+1)$-summations, that is

$$\sum_{i=1}^{2^s} \Sigma_i = \sum_{j=0}^{s} C_j \sum_{n=s}^{\infty} \frac{y^n}{(n-j)!} \exp(i(n+1)t) \tag{3.1}$$

where C_j are positive constants, depending upon s and j, not necessarily the same at each occurrence.

The proof of (3.1), in the case $s=0$, is obvious and, in the case $s=1$, is simple. Therefore we first take $s=2$, which gives the following 4-summations :

$$\sum_{n=2}^{\infty} \frac{y^n}{(n-2)!} \exp(i(n+1)t) + \sum_{n=2}^{\infty} \frac{y^n}{(n-2)!\,n} \exp(i(n+1)t)$$

$$+2\sum_{n=2}^{\infty} \frac{y^n}{(n-1)!} \exp(i(n+1)t) + 2!\sum_{n=2}^{\infty} \frac{y^n}{n!} \exp(i(n+1)t)$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \text{ say.}$$

And, since

$$\frac{1}{(n-2)!\,n} = \frac{1}{(n-1)!} - \frac{1}{n!},$$

we have

$$\sum_{i=1}^{4} \Sigma_i = \sum_{j=0}^{2} C_j \sum_{n=2}^{\infty} \frac{y^n}{(n-j)!} \exp(i(n+1)t),$$

where $C_0=1$, $C_1=3$ and $C_2=1$. Now, consider the case $s=3$ which gives the following summations :

$$\sum_{n=3}^{\infty}\frac{y^n}{(n-3)\,!}\exp\,(i\,(n+1)\,t)\;+\;\sum_{n=3}^{\infty}\frac{y^n}{(n-3)\,!}\left(\frac{1}{n}+\frac{2}{n-1}+\frac{3}{n-2}\right)\exp\,(i(n+1)\,t)$$

$$+\sum_{n=3}^{\infty}\frac{y^n}{(n-3)\,!}\left(\frac{2}{n(n-1)}+\frac{3}{n(n-2)}+\frac{6}{(n-1)\,(n-2)}\right)\exp\,(i\,(n+1)\,t)$$

$$+\sum_{n=3}^{\infty}\frac{y^n}{(n-3)\,!}\;\frac{3\,!}{n(n-1)\,(n-2)}\exp\,(i\,(n+1)\;t)$$

$$=\sum_{i=1}^{2^3}\;\Sigma_i\;,\quad\text{say.}$$

Again, using

$$\frac{1}{(n-3)\,!\;n}=\frac{1}{(n-2)\,!}-\frac{2}{(n-1)\,!}+\frac{2}{n\,!}$$

$$\frac{2}{(n-3)\,!\;(n-1)}=2\left(\frac{1}{(n-2)\,!}-\frac{1}{(n-1)\,!}\right)$$

$$\frac{6}{(n-3)\,!\;(n-2)}=\frac{6}{(n-2)\,!}$$

$$\frac{2}{(n-3)\,!\;(n-1)\,n}=2\left(\frac{1}{(n-1)\,!}-\frac{2}{n\,!}\right)$$

$$\frac{3}{(n-3)\,!\;(n-2)n}=3\left(\frac{1}{(n-1)\,!}-\frac{1}{n\,!}\right)$$

$$\frac{6}{(n-3\,!\;\;(n-2)\;\;(n-1)}=\frac{6}{(n-1)!}$$

and

$$\frac{3\,!}{(n-3)\,!\,(n-2)\,(n-1)\,n}=\frac{3\,!}{n\,!}$$

we have

$$\sum_{i=1}^{2^3}\;\Sigma_i=\sum_{j=0}^{3}\;C_j\sum_{n=3}^{\infty}\;\frac{y^n}{(n-j)\,!}\exp\,(i\,(n+1)\,t)$$

where $C_0=1$, $C_1=7$, $C_2=9$ and $C_3=1$.

Proceeding like this we shall get the general case (3.1) for $s\geq 0$, which will terminate the proof of Lemma 2.

4. Proof of the Theorems

Proof of Theorem 1—We have

$$A_n\;(x)=\frac{2}{\pi}\int_0^{\pi}\phi\,(t)\cos nt\;dt.$$

Integrating by parts, α-times, we have

$$A_n(x)=\frac{2}{\pi}\sum_{s=0}^{\alpha-1}\frac{(-1)^s}{\Gamma(s+1)}\pi^{s+1}\phi_{s+1}(\pi)(\cos n\pi)_s$$

$$+\frac{2}{\pi}\frac{(-1)^\alpha}{\Gamma(\alpha+1)}\int_0^\pi t^\alpha\phi_\alpha(t)(\cos nt)_\alpha\,dt$$

$$=\Sigma+K\,I, \text{ say,}$$

where

$$K=\frac{2}{\pi}\frac{(-1)^\alpha}{\Gamma(\alpha+1)}$$

and

$$I=\int_0^\pi t^\alpha\,\phi_\alpha(t)(\cos\ nt)_\alpha\,dt.$$

Since, by the hypothesis of Theorem 1,

$$t^{-\alpha}\,\phi_\alpha(t)\in BV(0,\pi)$$

we can (Carslaw 1930, p. 83) write

$$t^{-\alpha}\,\phi_\alpha(t)=y(t)-z(t)$$

where $y(t)$ and $z(t)$ are positive, monotonic increasing and bounded functions of t in $(0,\pi)$. And therefore, we have

$$I=\int_0^\pi t^{-\alpha}\,\phi_\alpha(t)\,t^{2\alpha}(\cos nt)_\alpha\,dt$$

$$=\int_0^\pi y(t)\,t^{2\alpha}\,(\cos nt)_\alpha\,dt-\int_0^\pi z(t)\,t^{2\alpha}(\cos nt)_\alpha\,dt.$$

Now, by using the second mean value theorem, have

$$I=y(\pi)\int_0^\pi t^{2\alpha}(\cos nt)_\alpha\,dt+(y(0)-y(\pi))\int_0^\eta t^{2\alpha}(\cos nt)_\alpha\,dt$$

$$(0\leq\eta\leq\pi)$$

$$-z(\pi)\int_0^\pi t^{2\alpha}(\cos nt)_\alpha\,dt$$

$$+(z(\pi)-z(0))\int_0^{\eta'} t^{2\alpha}\ (\cos nt)_\alpha\,dt$$

$$(0\leq\eta'\leq\pi)$$

$$= (y(\pi) - z(\pi)) \int_0^{\pi} t^{2a} \cos nt)_a \, dt$$

$$+ (y(0) - y(\pi)) \int_0^{\eta} t^{2a} (\cos nt)_a \, dt$$

$$(0 \leq \eta \leq \pi)$$

$$+ (z(\pi) - z(0)) \int_0^{\eta'} t^{2a} (\cos nt)_a \, dt.$$

$$(0 \leq \eta' \leq \pi)$$

Therefore the series $\sum_{n=1}^{\infty} A_n(x) \in |B|$, if

$$\int_0^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} (\cos (n+1) \pi)_s \right| dy < \infty \tag{4.1}$$

where s is an even integer such that $0 \leq s \leq \alpha - 1$. And

$$\int_0^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} \int_0^t u^{2a} (\cos (n+1) u)_a \, du \right| dy = O(1) \tag{4.2}$$

uniformly in $0 < t \leq \pi$.

It may be observed that the proof of (4.1) is contained in (4.2). Thus we only prove (4.2).

Proof of (4.2)—Integrating, 2α-times, by parts, we have

$$\int_0^t u^{2a} (\cos nu)_a \, du = \sum_{s=0}^{a-1} (-1)^s (\cos nt)_{a-s-1} (t^{2a})_s$$

$$+ \sum_{s=0}^{a-1} (-1)^{s+a} (\cos nt)_{-(1+s)} (t^{2a})_{s+a}$$

$$+ \int_0^t (u^{2a})_{2a} (\cos nu)_{-a} \, du$$

$$= \sum_{s=0}^{a-1} k (\cos nt)_s \, t^{1+s+a}$$

$$+ \sum_{s=0}^{a-1} k \, t^{a-s} n^{-(1+s)} . \begin{cases} \cos nt \, ; & s \text{ is odd} \\ \sin nt \, ; & s \text{ is even} \end{cases}$$

$$+ O\left\{n^{-(1+a)}\right\}$$

where k's denote the constants depending upon s and α and not necessarily the same at each occurrence. Therefore for the proof of (4.2) we only require to prove the following ;

$$\int_0^\infty e^{-y} \left| \sum_{n=0}^\infty \frac{y^n}{n!} (n+1)^{-(1+a)} \right| dy < \infty . \tag{4.2a}$$

$$I_1 = \int_0^\infty e^{-y} \left| \sum_{n=0}^\infty \frac{y^n}{n!} (\cos (n+1) t)_s \right| dy = O(t^{-1-s-a}) \tag{4.2b}$$

uniformly in $0 < t \leq \pi$ and s is an integer such that $0 \leq s \leq \alpha - 1$.

And

$$I_2 = \int_0^\infty e^{-y} \left| \sum_{n=0}^\infty \frac{y^n}{n!} \frac{J(nt)}{(n+1)^{1+s}} \right| dy = O(t^{s-a}) \tag{4.2c}$$

uniformly in $0 < t \leq \pi$, where $J(nt)$ is $\cos (n+1) t$ or $\sin (n+1) t$ according as s is odd or even integer such that $0 \leq s \leq \alpha - 1$.

The proof of (4.2*a*) follows from Lemma 1, since

$$\sum_{n=0}^\infty (n+1)^{-(1+a)}$$

is absolutely convergent.

Proof of (4,2*b*)—Now

$$I_1 = \int_1^\infty e^{-y} \left| \sum_{n=0}^\infty \frac{y^n}{n!} (n+1)^s \cdot \begin{Bmatrix} \cos (n+1) t \ ; \ s \text{ is even} \\ \sin (n+1) t \ ; \ s \text{ is odd} \end{Bmatrix} \right| dy + O(1)$$

$$= \int_1^\infty e^{-y} \left| \sum_{n=s}^\infty \frac{y^n}{n!} (n+1)^s \cdot \begin{Bmatrix} \cos (n+1) t \ ; \ s \text{ is even} \\ \sin (n+1) t \ ; \ s \text{ is odd} \end{Bmatrix} \right| dy + O(1).$$

Therefore, to prove (4.2*b*), it is sufficient to prove that

$$J = \int_1^\infty e^{-y} \left| \sum_{n=s}^\infty \frac{y^n}{n!} (n+1)^s \exp (i (n+1) t) \right| dy = O(t^{-1-s-a}),$$

uniformly in $0 < t \leqslant \pi$.

Now, by Lemma 2, for $0 \leqslant s \leq \alpha - 1$, we have

$$J = \int_1^\infty e^{-y} \left| \sum_{j=0}^s C_j y^j \exp (i (j+1) t) \sum_{n=s-j}^\infty \frac{y^n}{n!} (\exp (it))^n \right| dy$$

$$\leq \int_1^\infty e^{-y} \left| \sum_{j=0}^s C_j y^j \exp (i (j+1) t) \sum_{n=0}^\infty \frac{y^n}{n!} (\exp (it))^n \right| dy$$

$$+\int_1^\infty e^{-y}\left\{\sum_{j=0}^{s} C_j\, y^j \left| \exp\left(i\,(j+1)\,t\right) \right| \sum_{n=0}^{s-j-1} \frac{y^n}{n\,!} \left| \exp\,(int) \right|\right\} dy$$

$$\leqslant \int_1^\infty e^{-y}\left\{\sum_{j=0}^{\infty} C_j\, y^j \left| \exp\,(y \exp\,(it)\,) \right|\right\} dy$$

$$+\int_1^\infty e^{-y}\left\{\sum_{j=0}^{s} C_j\, O\,(y^{s-1})\right\} dy$$

$$= O\left\{\int_1^\infty y^s\, e^{-y+y\cos t}\, dy\right\} + O\left\{\int_1^\infty y^{s-1}\, e^{-y}\, dy\right\}$$

$$= O\left\{\int_0^\infty y^s\, e^{-(2\sin^2\frac{1}{2}t)\,y}\, dy\right\} + O\,(1)$$

$$= O\left\{(2\sin^2\tfrac{1}{2}\,t)^{-(s+1)}\;\Gamma\,(s+1)\right\} + O\,(1)$$

$$= O\,(t^{-2-2s}) + O\,(1)$$

$$= O\,(t^{-1-s-\alpha}),$$

uniformly in $0 < t \leq \pi$.

This completes the proof of (4.2) (*ii*).

Proof of (4.2*c*)—First we consider the case $0 \leq s \leq \alpha - 1$. In this case

$$\frac{J\,(nt)}{(n+1)^{1+s}} = O\{(n+1)^{-1-s}\}$$

and the uniform convergence of I_2, in $0 < t \leq \pi$, immediately follows by Lemma 1.

Now consider the case $s = 0$. We have

$$I_2 = \int_1^\infty e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{(n+1)\,!} \sin\,(n+1)\,t \right| dy + O(1)$$

$$= \int_1^\infty e^{-y}\, y^{-1} \left| \sum_{n=0}^{\infty} \frac{y^n}{n\,!} \sin nt \right| dy + O(1)$$

$$= \int_1^\infty e^{-y}\, y^{-1}\, e^{y\cos t} \mid \sin\,(y \sin t) \mid dy + O(1)$$

$$= O\left\{\int_1^\infty y^{-1}\, e^{y(\cos t-1)}\, dy\right\} + O(1).$$

And

$$L = \int_1^\infty y^{-1}\, e^{y(\cos t-1)}\, dy$$

$$= \left(\int_1^{t^{-2}} + \int_{t^{-2}}^{\infty} \right) \left(y^{-1}\, e^{y(\cos t-1)}\, dy \right)$$

$$= O\left\{ \int_1^{t^{-2}} y^{-1}\, dy \right\} + \int_{t^{-2}}^{\infty} y^{-1}\, e^{-(2 \sin^2 1/2\, t)y}\, dy$$

$$= O\left(\log \frac{2\pi}{t} \right) + O\left\{ t^2 \int_1^\infty e^{-2(\sin^2 1/2 t)y}\, dy \right\}$$

$$= O\left(\log \frac{2\pi}{t} \right) + O(1)$$

$$= O\left(\log \frac{2\pi}{t} \right)$$

uniformly in $0 < t \leq \pi$.

On substituting the order-estimate for L in I_2 $(s=0)$ and combining the results we get, for $0 \leq s \leq \alpha - 1$;

$$I_2 = O\left(\log \frac{2\pi}{t} \right)$$

and hence we follow the proof of (4.2c).

This completes the proof of (4.2), which, incidently, terminates the proof of Theorem 1.

Proof of Theorem 2—We have

$$B_n(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \sin nt\, dt.$$

Proceeding as in Theorem 1, we have

$$B_n(x) \doteq \frac{2}{\pi} \sum_{s=0}^{\alpha-1} \frac{(-1)^s}{\Gamma(s+1)}\, \pi^{s+1}\, \psi_{s+1}(\pi)\, (\sin n\pi)_s$$

$$+ \frac{2}{\pi}\, \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \left[(y_1(\pi) - z_1(\pi)) \int_0^\pi t^{2\alpha} (\sin nt)_\alpha\, dt \right.$$

$$+ (y_1(0) - y_1(\pi)) \int_0^\eta t^{2\alpha} (\sin)\ nt_\alpha\, dt \qquad (0 \leqslant \eta \leqslant \pi)$$

$$\left. + (z_1(\pi) - z_1(0)) \int_0^{\eta'} t^{2\alpha} (\sin\ nt)_\alpha\, dt \right] , \qquad (0 \leqslant \eta' \leqslant \pi)$$

where $z_1(t)$ and $y_1(t)$ are positive, monotonic increasing and bounded functions of t in $(0, \pi)$ and such that

$$t^{-\alpha}\,\psi_\alpha(t) = y_1(t) - z_1(t).$$

Again, proceeding as in Theorem 1,

$$\sum_{n=1}^{\infty} B_n(x) \in |B|, \text{ if}$$

$$\int_0^\infty e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} (n+1)^{-1-\alpha} \right| dy < \infty \tag{4.3}$$

$$\int_0^\infty e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{\bar{K}(nt)}{(n+1)^{1+s}} \right| dy = O(t^{s-\alpha}) \tag{4.4}$$

uniformly in $0 < t \leq \pi$, where $\bar{K}(nt)$ is either $\sin(n+1)t$ or $\cos(n+1)t$ according as s is odd or even integer such that $0 \leq s \leq \alpha-1$.

$$\int_0^\infty e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} (\sin(n+1)\pi)_s \right| dy < \infty \tag{4,5}$$

where s is odd integer and such that $1 \leqslant s \leq \alpha-1$.

$$\int_0^\infty e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!} (\sin(n+1)t)_s \right| dy = O(t^{-1-s-\alpha}) \tag{4.6}$$

uniformly in $0 < t \leq \pi$ and s is any integer such that $0 \leq s \leq \alpha-1$.

The proof of (4.3) follows from Lemma 1. The proof of (4.4) runs parallel to that of (4.2*c*) by using the order-estimate for L. Essentially the proof of (4.5) is contained in (4.6) for $t=\pi$ and the proof of (4.6) parallel to that of (4.2*b*).

This terminates the proof of Theorem 2.

References

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