ON GENERALIZATION OF BANACH'S FIXED POINT THEOREM

by SUDHANSHU KUMAR GHOSHAL *and* BARADA K. RAY,
Dapartment of Mathematics, Regional Engineering College, Durgapur 9

(Communicated by F. C. Auluck, F. N. A.)

(*Received 7 September 1972*)

We have shown in this paper that if T be a continuous self-mapping of a metric space (X, ρ) such that (*i*) $\rho(Tx, Ty) < k_1 \rho(x, Tx) + k_2 \rho(y, Ty)$ for each $x\ y \in X$, $k_1 > 0$, $k_2 > 0$, $k_1 + k_2 < 1$ (*ii*) there exists a subset $M \subset X$ and a point $x_0 \in M$ such that $\rho(x, x_0) - \rho(Tx, Tx_0) \geqslant (1+k)\ \rho(x_0, Tx_0)$ for every $x \in X - M$, $k = \frac{k_1}{1-k_2}$ and (*iii*) T maps M into a compact subset of X then T has a uniqe fixed point.

Also if T is a continuous self-mapping of a complete metric space (X, ρ) satisfying (*i*) and (*ii*) and

$\rho(Tx, Ty) \leqslant \alpha(\rho(x, y))\ \rho(x, Tx) + \beta(\rho(x, y))\ \rho(y, Ty)$

for every $x, y \in M$, where $0 \leqslant \alpha(\rho) < 1$, $0 \leqslant \beta(\rho) < 1$ for every $\rho > 0$, $\alpha(\rho) + \beta(\rho) < 1$ and $\alpha(\rho)$, $\beta(\rho)$ are monotonically decreasing functions of ρ, then T has a unique fixed point.

A mapping f of a metric spacc X into itself is called a contraction mapping if the condition $\rho(f(p), f(q)) \leq \lambda \rho(p, q)$ with the constant λ, $0 \leq \lambda < 1$, holds for every $p, q \in X$.

The well known Banach (1922) contraction principle states that a contraction mapping on a complete Metric space X has a unique fixed point.

The mapping f of a metric space X into itself is called globally contractive if the condition $\rho(f(x), f(y)) < \lambda \rho(x, y)$ with constant λ, $0 \leqslant \lambda < 1$ holds for every $x, y \in X$, $x \neq y$

Rakotch (1962) has generalized Banach's contraction principle by replacing λ with a function $\lambda\ (x, y)$ by suitably defining the family of functions $\{\lambda(x, y)\}$. In this paper we use the idea of generalized contraction mapping as introduced by Rakotch (1962) and a few theorems on sequence of mappings and common fixed points have been proved. Throughout this paper X will denote a metric space (unless otherwise stated) and ρ the metric on X.

Definition :—Let F denote the family of functions $\lambda(x, y)$ satisfying the following conditions

(*i*) $\lambda(x, y) = \lambda(\rho(x, y))$, i. e. λ depends on the distance between x and y only.

(*ii*) $0 \leqslant \lambda(\rho) < 1$ for every $\rho > 0$.

(*iii*) $\lambda(\rho(x, y))$ is a monotonically decreasing function of $\rho(x, y)$.

Theorem 1—Let f be a continuous mapping of X into itself such that $\rho(f(x), f(y)) < \alpha\, \rho(x, f(x)) + \beta\, \rho(y, f(y))$ for $x, y \in X$, $x \neq y$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$.

If for some $x_0 \in X$ the sequence $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ converging to a point $\xi \in X$, then ξ is the unique fixed point of f.

Proof :—Let $x_0 \in X$ be arbitrary and let us define the sequence of elements $\{x_n\}$ as $x_n = f^n(x_0)$, $x_{n+1} = f(x_n)$, $n=0, 1, 2, \ldots\ldots$

It can be easily seen that the sequence $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to ξ we conclude that $\{x_n\}$ also converges to ξ,

$$\text{i. e. } \lim_{n \to \infty} x_n = \xi.$$

Since f is continuous, $f(\xi) = f \lim x_n = \lim f(x_n) = \lim x_{n+1} = \xi$.

If possible let ξ and η be two fixed points such that $\xi \neq \eta$.

So $\rho(\xi, \eta) > 0$.

Now $\rho(\xi, n) = \rho(f(\xi), f(n)) < \alpha\, \rho(\xi, f(\xi)) + \beta\, \rho(\eta, f(\eta)) = 0$, which is impossible.

Hence ξ is unique.

Theorem 2. Let f be a continuous mapping of X into itself such that $\rho(f(x), f(y)) < \alpha\, \rho(x, f(x)) + \beta\, \rho(y, f(y))$

for $x, y \in X, x \neq y$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$.

Further, let there exist a subset $M \subset X$ and a point $x_0 \in M$ such that $\rho(x, x_0) - \rho(f(x), f(x_0)) \geqslant (1+k)\, \rho(x_0 . f(x_0)) \ldots (A)$

for every $x \in X - M$, where $k = \dfrac{\alpha}{1-\beta} < 1$ and that f maps M into a compact subset of X. Then there exists a unique fixed point of f.

Proof: Let $x_0 \neq f(x_0)$ and define $x_n = f^n \ (x_0)$,

$$x_{n+1} = f(x_n). \quad n=0,\ 1,\ 2, \ldots\ldots .$$

Since f maps M into a campact set, we shall show that $x_n \in M$ for every n and the rest will follow as a direct consquence of Theorem 1.

It has been noted in Theorem 1 that $\{x_n\}$ is a Cauchy sequence and it is easy to see that $\rho\ (x_n,\ x_{n+1}) < \left(\frac{\alpha}{1-\beta}\right)^n \rho\ (x_0,\ f(x_0))$

$$< \left(\frac{\alpha}{1-\beta}\right) \rho\ (x_0, f\ (x_0)).$$

So

$$\rho\ (x_n,\ x_0) \leqslant \rho\ (x_0, x_1) + \rho\ (x_1, x_{n+1}) + \rho\ (x_n,\ x_{n+1})$$

$$< \rho\ (x_0, (f(x_0)) + \rho\ (f(x_0), (f(x_n)) + k\ \rho\ (x_0, f\ (x_0)).$$

Therefore

$$\rho\ (x_n,\ x_0) - \rho\ (f(x_n), f(x_0)) < (1+k)\ \rho\ (x_0, f(x_0))$$

and so from (A) it follows that $x_n \in M$ for every n.

Corollary to Theorem 2—Let f be a continuous mapping of X into itself satisfying

$$\rho\ (f(x),\ f(y)) < k_1\ \rho\ (x, f(x)) + k_2\ \rho\ (y, f(y))\ k_1 > 0,\ k_2 > 0,$$
$$k_1 + k_2 < 1$$

for every $x,\ y \in X,\ x \neq y$ and let there exist a point $x_0 \in X$ such that

$$\rho\ (f(x), f(x_0)) \leq \alpha\ (x,\ x_0)\ \rho\ (x,\ x_0)$$

for every $x \in X$, where $\alpha\ (x.\ x_0) \in F$ and that f maps

$$S\ (x_0,\ r) = \{x \mid \rho\ (x,\ x_0) < r\}$$

with

$$r = \frac{(1+k)\ \rho\ (x_0, f(x_0))}{1-\alpha\ [(1+k)\ \rho\ (x_0, f(x_0))]}$$

$$k = \frac{k_1}{1-k_2}$$

into a compact subset of X, then there exists uniqe fixed point of f.

Proof: In Theorem 2, let us take $M = S\ (x_0,\ r)$. Then, since

$$\alpha\ (x,\ x_0) = \alpha\ (\rho\ (x_0,\ x)) \in F,\ \alpha\ (\rho)$$

is monotone decreasing ; and since $r \geq (1+k)\ \rho\ (x_0, f(x_0))$

we have, [from the given condition,]

$$\rho(x, x_0) - \rho(f(x), f(x_0)) \geqslant \rho(x, x_0) - \alpha(x, x_0)\, \rho(x, x_0)$$
$$=[1 - \alpha(x, x_0)]\, \rho(x, x_0).$$

If $\rho(x, x_0) \geq r$, i. e. $x \notin M$ then

$$\rho(x, x_0)-\rho(f(x), f(x_0)) \geq [1-\alpha(r)]\, r \geq \{1-\alpha\{(1+k)\,\rho(x_0, f(x_0))\}]\, r$$
$$=(1+k)\,\rho(x_0, f(x_0)).$$

So the condition (A) of Theorem 2 holds. Hence etc.

Theorem 3—If f be a continuous mapping of a complete metric space X into itself such that

$$\rho(f(x), f(y) < k_1\, \rho(x, f(x))+k_2\, \rho(y, f(y))$$

for all $x, y \in X$, $x \neq y$, $k_1 > 0, k_2 > 0$, $k_1+k_2 < 1$

and that there exists a subset $M \subset X$ and a point $x_0 \in M$ satisfying

(*i*) $\rho(x, x_0)-\rho(f(x), f(x_0)) \geq (1+k)\,\rho(x_0, f(x_0))$,

$$k = \frac{k_1}{1-k_2} < 1$$

for every $x \in X-M$,

(*ii*) $\rho(f(x), f(y)) \leqslant \alpha(x, y)\, \rho(x, f(x)) + \beta(x, y)\, \rho(y, f(y))$

for every $x, y \in M$, $\alpha(x, y) \in F$, $\beta(x, y) \in F$ and that $\alpha(\rho)+\beta(\rho) < 1$ for every $\rho > 0$, then there exists a unique fixed point of f.

Proof: Let $x_0 \neq f(x_0)$ and let $x_1 = f(x_0)$, $x_2 = f(x_1), \ldots$

$x_n = f^n(x_0)$, $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$

we have $\rho(x_n, x_{n+1}) < k\, \rho(x_0, f(x_0))$

and that $x_n \in M$ for every n.

We show that the sequence $\{x_n\}$ is bounded, i.e., the set of numbers $\rho(x_n, \theta)$ is bounded for every fixed point θ of the space (Lusternik and Sobolen 1961). Now

$$\rho(x_0, x_n) \leqslant \rho(x_0, f(x_0)) + \rho(f(x_0), f(x_n))$$
$$+ \rho(x_{n+1}, x_n) < \rho(x_0, f(x_0)) + \rho(f(x_0), f(x_n))$$
$$+ k\, \rho(x_0, f(x_0) \leqslant \rho(x_0, f(x_0)) + \alpha(x_0, x_n)\, \rho(x_0, f(x_0))$$
$$+ \beta(x_0, x_n)\, \rho(x_n, x_{n+1}) + k\, \rho(x_0 f(x_0)) < \rho(x_0, f(x_0))$$
$$+ \alpha(x_0, x_n)\, \rho(x_0, f(x_0) + \beta(x_0, x_n)\, k\, \rho(x_0, f(x_0))$$
$$+ k\, \rho(x_0, f(x_0))$$
$$= [1 + k + \alpha(x_0, x_n + k\, \beta(x_0, (x_0, x_n)]\, \rho(x_0, f(x_0)).$$

If $\rho(x_0, x_n) \geq \rho'$ for a given $\rho' > 0$, then since $\alpha(\rho)$ and $\beta(\rho)$ are monotone decreasing so $\rho(x_0, x_n) \leqslant \rho(x_0, f(x_0))\,[1+k+\alpha(\rho')+k\,\beta(\rho')] = R_1$.
Hence $\rho(x_0, x_n) \leq R$ where $R = \max(\rho', R_1)$, i. e. the sequence $\{x_n\}$ is bounded, $n = 1, 2, \ldots$.

Next we wish to prove that $\{x_n\}$ is a Cauchy sequence.

Now $\rho(x_1, x_2) = \rho(f(x_0), f(x_1)) \leqslant \alpha(x_0, x_1)\,\rho(x_0, x_1) + \beta(x_0, x_1)\,\rho(x_1, x_2)$.
So

$$\rho(x_1, x_2) \leq \frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)}\,\rho(x_0, x_1)$$

$$\rho(x_2, x_3) = \rho(f(x_1), f(x_2)) \leqslant \alpha(x_1, x_2)\,\rho(x_1, x_2) + \beta(x_1, x_2)\,\rho(x_2, x_3)$$

$$\therefore\ \rho(x_2, x_3) \leq \frac{\alpha(x_1, x_2)}{1-\beta(x_1, x_2)}\,\rho(x_1, x_2)$$

$$\leq \frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)}\;\frac{\alpha(x_1, x_2)}{1-\beta(x_1, x_2)}\,\rho(x_0, x_1)\ \ldots.$$

$$\rho(x_n, x_{n+1}) \leqslant \frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)}\;\frac{\alpha(x_1, x_2)}{1-\beta(x_1, x_2)}\ \ldots.$$

$$\frac{\alpha(x_{n-1}, x_n)}{1-\beta(x_{n-1}, x_n)}\,\rho(x_0\ x_1).$$

Let $p > 0$ be an arbitrary integer.

Then $\rho(x_n, x_{n+p}) \leqslant \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p})$

$$\leq \Bigg[\frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)} \cdots \frac{\alpha(x_{n-1}, x_n)}{1-\beta(x_{n-1}, x_n)} + \frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)} \cdots$$

$$\frac{\alpha(x_n, x_{n+1})}{1-\beta(x_n, x_{n+1})} + \ldots + \ldots + p\text{th term}\Bigg]\,\rho(x_0, x_1).$$

It will be sufficient to show that for every $\epsilon > 0$ there exists a number N which depends on ϵ (and not on p) such that for every $p > 0$, $\rho(x_N, x_{N+p}) < \epsilon$ (since sequence $\rho(x_n, x_{n+p})$ is a decreasing sequence).

So let $\epsilon > 0$ be arbitrary.

If $\rho(x_i, x_{i+1}) \geq \epsilon$ for $i=0, 1, 2, \ldots, n+p-1$ then since $\alpha(\rho), \beta(\rho)$ are monotonic decreasing for every $\rho > 0$ and since $\alpha(\rho)+\beta(\rho) < 1, \rho > 0$,

we have

$$\alpha(x_i, x_{i+1}) \leqslant \alpha(\epsilon),\ \beta(x_i, x_{i+1}) \leqslant \beta(\epsilon)$$

and

$\alpha(\epsilon) + \beta(\epsilon) < 1$, i. e. $\lambda(\epsilon) = \dfrac{\alpha(\epsilon)}{1-\beta(\epsilon)} < 1.$

Hence

$$\rho\,(x_n, x_{n+p}) \leq [\lambda\,(\epsilon)\,]^n\,[1+\lambda\,(\epsilon) + \{\lambda\,(\epsilon)\,\}^2 + \ldots + \ldots \{\lambda\,(\epsilon)\,\}^{p-1}]\,\rho\,(x_0, x_1)$$

$$< \frac{R\,[\lambda\,(\epsilon)\,]^n}{1-\lambda\,(\epsilon)} \quad \to 0 \text{ as } n \to \infty.$$

Hence there exists an integer N independent of p, such that

$$\rho\,(x_N, x_{N+p}) < \epsilon \text{ for every } p > 0.$$

So $\{x_n\}$ is a Cauchy sequence.

Now since X is complete $\xi = \lim x_n \in X$ and since f is continuous we have

$$f\,(\xi) = f \lim x_n = \lim f\,(x_n) = \lim x_{n+1} = \xi.$$

Thus ξ is a fixed point of f.

As before it can be proved that ξ is uniqe.

Taking $M = X$ we get the corollary.

Corollary—If for every pair of points x, y belonging to a complete metric space X we have

$$\rho\,(\,f\,(x), f(y)\,) \leq \alpha\,(x, y)\;\rho\,(x, f\,(x)\,) + \beta\,(x, y)\;\rho\,(y, f\,(y)\,)$$

where

$$\alpha\,(x, y) \in F,\ \beta\,(x, y) \in F \text{ and } \alpha\,(\rho) + \beta\,(\rho) < 1$$

for every $\rho > 0$, then there exists a unique fixed point f.

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