ON WEAKENED FIELD EQUATIONS IN GENERAL RELATIVITY

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(Communicated by F.C. Auluck, F.N.A.)

(Received 4 February 1974)

Certain weakened field equations which have been suggested as alternative to the vacuum field equations of general relativity are investigated for their solutions in a four-dimensional Riemannian space with the line element $ds^2 = g_{ij}\, dx^i dx^j$. It is shown that the Ricci tensor resembles as electromagnetic field tensor when charge current vector is zero while satisfying the equation due to Rund (1967). Also a plane wave solution in the sense of Takeno (1961) is given.

1. INTRODUCTION

We have considered a four-dimensional Riemanian space with the line element

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1{\cdot}1)$$

where it is assumed that

$$g \equiv \det|\, g_{ij}\,| < 0.$$

We shall recall certain well known definitions and results (Synge and Schild 1961). Thc Ehristoffel symbol of second kind are defined by

$$\left\{ {i \atop jk} \right\} = (1/2) g^{ih} \,(\partial x^k\, g_{gh} + \partial x^j\, g_{kh} - \partial x^h\, g_{jk}), \quad (\partial x^i \equiv \partial/\partial x^i\,)$$

where the quantities g^{ij} are the elements of the inverse matrix of g_{ij}. The curvature tensor $R^j{}_{ikl}$ given by

$$R^j{}_{ikl} = \partial x^l \left\{ {j \atop ik} \right\} - \partial x^k \left\{ {j \atop il} \right\} + \left\{ {h \atop ik} \right\} \left\{ {j \atop hl} \right\} - \left\{ {h \atop il} \right\} \left\{ {j \atop kh} \right\}$$

satisfies

$$R_{ihkl} = g_{jh}\, R^j{}_{ikl}\,.$$

The Ricci tensor R_{ij} and the scalar curvature R are defined by

$$R_{ij} \;= R^h{}_{ijh}$$

and

$$R \quad = g^{ij}\; R_{ij}$$

respectively. Furthermore, $R^j{}_{ikl}$ satisfies the Bianchi identities

$$R^j{}_{ikl;\,h} + R^j{}_{ihk;\,l} + R^j{}_{ilh;\,k} = 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1{\cdot}2)$$

where a semi-colon followed by an index denotes covariant differentiation. From (1·2) it follows

$$R^{j}{}_{ikl;j} = R_{ik;l} - R_{il;k} \qquad \ldots \qquad (1\cdot3)$$

and

$$R^{i}{}_{j;i} = (1/2)\, R_{;j} \qquad \ldots \qquad (1\cdot4)$$

where $R^{i}{}_{j} = g^{ih}\, R_{hj}$.

The field equations of general relativity as proposed by Einstein are

$$G_{ij} = -kT_{ij} \qquad \ldots \qquad (1\cdot5)$$

where G_{ij} is the Einstein tensor

$$G_{ij} = R_{ij} - (1/2)\, g_{ij}\, R$$

k is a constant and T_{ij} is the energy-momentum tensor in presence of matter. G_{ij} enjoys the properties

(a) $G_{ij} = G_{ji}$ and (b) $G^{i}{}_{j;i} = (g^{hi}\, G_{hj}\,);i = 0.$

In vaccum T_{ij} vanishes and (1·5) reduces to

$$R_{ij} = 0. \qquad \ldots \qquad (1\cdot6)$$

These are taken as Einstein equations in vaccum and its solution together with geodesic hypothesis (the trajectories of test particles are geodesic) give agreement with experiment. Various authors have proposed that (1·6) be weakened by replacing them with various alternative vacuum field equations which admit (1·6) as a subclass of solutions. Such field equations have been called Weakened Field Equations have been called Weakened Field Equations (WFE). Here we shall consider some WFE and shall obtain their solutions in the space-time given by (1·1) for which the scalar curvature R is zero.

2. Weakened Field Equations

We shall consider here four vacuum field equations which have been suggested by Kilmister and Newman (1961), Pirani (see lovelock (1967*a*), Rund 1964) and Eddington as alternatives to the vacuum field equations of Einstein's theory of general relativity. They are respectively:

$$J_{ikl} \equiv R^{j}_{ikl;j} = 0 \qquad \ldots \qquad (2\cdot1)$$

$$G_{jk} \equiv (-g)^{1/4}\,[\,g^{ih}R_{kj;ih} - g^{ih}\, R_{ij;kh} + (1/6)R_{;kj} - (1/6)g_{jk}\, g^{ih}\, R_{;ih} - R^{ih}\, C_{jhik} + (R/6)g^{ih}\, C_{jhik}] = 0 \qquad \ldots \qquad (2\cdot2)$$

with properties

(a) $G_{jk} = G_{kj}$ and (b) $G^{j}_{k;j} = 0,$

$$E^{hk} \equiv (-g)^{1/2} [g^{hj} g^{ki} \{ 2R_{jlim} R^{ml} + g^{ml} R_{ij;lm} - R_{;ij}\} - (1/2) g^{hk} \{ R^l_m R^m_l - g^{lm} R_{;lm} \}] = 0 \quad (2\cdot3)$$

with properties

$$\text{(a) } E^{hk} = E^{kh} \quad \text{and} \quad \text{(b) } E^{hk}_{;h} = 0,$$

and

$$\mathscr{E}^{rs} \equiv (-g)^{1/2} [(g^{rs}g^{tu} - (1/2)g^{rt}g^{su} - (1/2)g^{ru}g^{st})R_{;ut} + R(R^{sr} - (1/4)g^{sr}R)] = 0 \quad (2\cdot4)$$

with properties

$$\mathscr{E}^{rs} = \mathscr{E}^{rs} \quad \text{and (b) } \mathscr{E}^{rs}_{;r} = 0,$$

where C_{jhlk} is Weyl conform tensor given by

$$C_{jhik} = R_{jhik} - (1/2) (R_{ji} g_{hk} - R_{hi} g_{jk} - R_{jk} g_{hi} + R_{hk} g_{ij}) + (R/6) (g_{ij} g_{hk} - g_{hi} g_{jk}). \quad (2\cdot5)$$

These are weaker than the Einstein equations in vacuum in the sense that they each admit a class of solutions for which

$$R_{ij} = 0$$

and hence they have been called WFE.

3. Solutions of WFE

We shall consider the space-time having scalar curvature zero (*i*) when it is conformally flat and (ii) when it is non-conformally flat.

Case I. When the space-time is conformally flat, the Weyl curvature vanishes, i.e. (2.5) gives

$$R_{jhik} = (1/2) (R_{ji} g_{hk} - R_{hi} g_{jh} - R_{jk} g_{hi} + R_{hk} g_{ij}) \quad (3\cdot1)$$

from which it follows that

$$R^j_{ihk;j} = (1/2) (R_{ih;k} - R_{ik;h}) \quad (3\cdot2)$$

by virtue of $R = 0$ and (1·4). Combining (1·3) and (3·2) we have

$$R^j_{ihk;} = 0 \quad (3\cdot3)$$

or, equivalently

$$R_{ih;k} = R_{ik;h}. \quad (3\cdot4)$$

Therefore WFE (2·2) reduces to

$$\left.\begin{aligned} \mathscr{G}_{jk} &\equiv (-g)^{1/4} [g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh}] = 0 \\ \text{or} \quad & (-g)^{1/4} g^{ih} [R_{kj;i} - R_{ij;k}]_{;h} = 0 \end{aligned}\right\} \quad (3\cdot5)$$

which is identically satisfied in view of (3·4). The equation (3·3) is WFE (2·1). The WFE (2·4) is satisfied by $R = 0$ alone. The WFE (2·3) reduces to

$$E^{hk} \equiv (-g)^{1/2} \left[g^{hj} g^{ki} \{ 2R_{jlim} R^{ml} + g^{ml} R_{ij;lm} \} - (1/2) g^{hk} R^l_m R^m_l \right] = 0. \quad (3\cdot6)$$

Multiplying (3·6) by g_{hk} we have

$$(-g)^{1/2} \left[-2R_{lm} R^{ml} + g^{ml} R_{;lm} - 2R^l_m R^m_l \right] = 0$$

which is satisfied if

$$R_{lm} R^{ml} + R^l_m R^m_l = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot7)$$

as $R = 0$ and $g \neq 0$. Hence we have

g_{ij} given by (1·1) representing a conformally flat space-time with scalar curvature zero is a solution of each of the WFE (2·1), (2·2) and (2·4). It is also a solution of WFE (2·3) if (3·7) holds.

Case II. When the space-time is non-conformally flat but $R = 0$, we have

$$C_{jhik} = R_{jhik} - (1/2)\,(R_{ji}\, g_{hk} - R_{ih}\, g_{jk} - R_{jk} g_{hi} + R_{hk}\, g_{ij}). \quad \ldots \quad (3\cdot8)$$

The WFE (2·1) in view of (1·3) becomes

$$R_{ik;l} - R_{il;k} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot9)$$

which on contraction gives

$$R^i_{j;i} = R_{;j}\,. \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot10)$$

Since $R = 0$ or combining (3·10) with (1·4) we find

$$R^i_{j;i} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot11)$$

The WFE (2·2) reduces to

$$\mathscr{G}_{jk} \equiv (-g)^{1/4} (g^{ih} R_{kj;ih} - g^{ih} R_{ij;kh} - R^{ih} C_{jhik}) = 0 \qquad \ldots \quad (3\cdot12)$$

as $R = 0$. Now multiplying (3·12) by g^{jk} and remembering that g_{ij} behaves as constant in covariant differentiation, we have

$$(-g)^{1/4} [g^{ih} R_{;ih} - g^{ih} (R^k_{i:k})_{;h} - R^{ih} \{3R_{hi} - (1/2)\,(R g_{hi} - \delta^i_h R_{ji} + \delta^k_i R_{hk})\}] = 0 \quad (3\cdot13)$$

where $\delta^i_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$. In view of $R = 0$ and (1·4) the equation (3·13) is satisfied if

$$R^{ih} R_{ih} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot14)$$

The WFE (2·3) for $R = 0$ becomes

$$E^{hk} = (-g)^{1/2} [g^{hj} g^{ki} (2R_{jlim} R^{ml} + g^{ml} R_{ij;lm}) - (1/2) g^{hk} R^l_m R^m_l] = 0. \quad (3\cdot15)$$

Contracting (3·15) (by multiplying g_{hk}) and making use of $R = 0$ and $g_{hk}\, g^{hk} = 4$, we find that it is satisfied when

$$R_{lm} R^{mi} + R^l_m R^m_l = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot16)$$

The WFE (2·4) is satisfied by $R = 0$ alone. Hence we have:

g_{ij} given by (1·1) representing a non-conformally flat space-time with scalar curvature zero is a solution of WFE (2·4). It is also a solution of WFE (2·1), (2·2) and (2·3) under the conditions (3·11), (3·14) and (3·16) respectively.

4. Note

Various authors have found the solutions of WFE in a hope to give an useful interpretation. Lovelock (1967*a*, *b*) has found the solutions in a spherically symmetric space-time which he proved to be gravitationally unphysical. Dolan (1968), Thompson (1963), Swami (1970) and many others have also worked in this direction. However, it follows easily from (1·4) and (3·10) that

$$R_{;j} = 0 \quad \ldots \quad (4·1)$$

which reduces the WFE (2·1) to (3·11) or

$$R^{i}_{j;i} = 0. \quad \ldots \quad (4·2)$$

Raising the index j we have

$$R^{ih}_{;i} = 0. \quad \ldots \quad (4·3)$$

The equation (4·3) is equivalent to the field equation of Rund (1967) given by

$$R^{ij}_{;k} = 0 \quad \ldots \quad (4·4)$$

if $i = k$ in (4·4). Also, the equation (4·3) can be taken as analogue to the Maxwell's equations

$$F^{ij}_{;j} = 0 \quad \ldots \quad (4·5)$$

and, therefore, the Ricci tensor resembles as electromagnetic field tensor $F_{ij} = -F_{ji}$ when charge current vector J^{μ} is zero.

5. Particular Case

In this section we shall find the plane wave solution (in the sense of Takeno (1961) of WFE. Therefore we take the space-time of Takeno (1961)

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - (C-E)dz^2 - 2Edzdt + (C+E)dt^2 \quad \ldots \quad (5·1)$$

where $A,B,C,D,$ are functions of $E = z-t$. For this metric

$$R = 0,\ R^{l}_{m} R^{m}_{l} = 0,\ R_{ml} R^{ml} = 0$$

$$\left.\begin{aligned} R_{33} &= -R_{34} = R_{44} = (BL-2DN+AM)/2m \text{ and other } R_{ij} = 0 \\ R^{33} &= R^{34} = R^{44} = (BL-2DN+AM)/2mC^2 \text{ and other } R^{ij} = 0 \\ R^{3}_{4} &= -R^{3}_{3} = -R^{4}_{3} = R^{4}_{4} = (BL-2DN+AM)/2mC \text{ and other } R^{i}_{j} = 0. \end{aligned}\right\} \quad (5·2)$$

Hence it can easily be seen that conditions (3·11), (3·14) and (3·16) are satisfied and the scalar curvature is zero. Obviously the metric (5·1) is a solution of $R_{ij} = 0$ if and only if

$$BL + AM = 2DN, \qquad \ldots \qquad (5{\cdot}3)$$

where

$$2L = \bar{\bar{A}}—\bar{A}\bar{C}/C—(B\bar{A}^2+A\bar{D}^2—2\bar{A}D\bar{D})/2m \qquad (— \equiv \partial/\partial z)$$

$$2M = \bar{\bar{B}}—\bar{B}\bar{C}/C—(A\bar{B}^2+B\bar{D}^2—2\bar{B}D)\bar{D}/2m$$

$$2N = \bar{\bar{D}}—\bar{D}\bar{C}/C(A\bar{B}\bar{D}+\bar{A}B\bar{D}—\bar{A}\bar{B}D—D\bar{D}^2)/2m$$

and $m = AB—D^2$.

Thus, the equations (2·1)—(2·4) which have (5·1) as a solution include the solution of $R_{ij} = 0$ under the condition (5·3), and hence they are WFE.

Acknowledgements

The author is grateful to Professor P.C. Vaidya and Professor K.B. Lal for their advice and encouragement.

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