# EXPANSIONS OF GENERALIZED $H$-FUNCTION

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(Communicated by J. N. Kapur, F. N. A.)

(*Received 21 September 1973, after revision 23 January 1974*)

In the present paper the authors have formulated some interesting expansions involving generalized $H$-function ; expansions have been obtained in terms of product of generalized $H$-function and hypergeometric function and product of generalized $H$-function and Fox's $H$-function, The unique result of this paper is a result in which expansion of Fox's $H$-function has been obtained as a product of two $H$-functions of different arguments. Some interesting particular cases have also been obtained.

## 1. INTRODUCTION

In the notation of Saxena (1971) the generalized $H$-function, defined earlier by Munot and Kalla (1971) and Verma (1971), is represented in the following manner :

$$H^{l,n,n_1,m,m_1}_{E,(A:C),F,(B:D)}\left[\begin{array}{c|c} x & (e,\theta) \\ & (a,\alpha);(c,\gamma) \\ y & (f,\phi) \\ & (b,\beta);(d,\delta) \end{array}\right]$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{L_1}\int_{L_2}\chi_1(\xi)\,\chi_2(\eta)\,\chi_3(\xi+\eta)\,x^{-\xi}\,y^{-\eta}\,d\xi\,d\eta \qquad (1.1)$$

where an empty product is interpreted as unity.
Here

$$\chi_1(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j+\xi\beta_j)\prod_{j=1}^{n}\Gamma(1-a_j-\xi\alpha_j)}{\prod_{j=m+1}^{B}\Gamma(1-b_j-\xi\beta_j)\prod_{j=n+1}^{A}\Gamma(a_j+\xi\alpha_j)}$$

$$\chi_2(\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(d_j-\eta\delta_j)\prod_{j=1}^{n_1}\Gamma(1-c_j-\eta\gamma_j)}{\prod_{j=m_1+1}^{D}\Gamma(1-d_j-\eta\delta_j)\prod_{j=n_1+1}^{C}\Gamma(c_j+\eta\gamma_j)}$$

$$\chi_3(\xi+\eta)=\frac{\prod_{j=1}^{l}\Gamma[e_j-(\xi+\eta)\theta_j]}{\prod_{j=l+1}^{E}\Gamma[1-e_j+(\xi+\eta)\theta_j]\prod_{j=1}^{F}\Gamma[f_j-(\xi+\eta)\theta_j]}$$

For a detailed account of this function and various remarks for its convergence can be seen in the paper of Saxena (1961), which is avoided here due to lack of space.

## 2. The Expansion Formulae

*Theorem* 2.1—We establish an expansion formula for the generalized $H$-function, expressed into a product of generalized $H$-function and hypergeometric function.

$$x^{\lambda} H \begin{bmatrix} ax \\ bx \end{bmatrix} = h \frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{q} \Gamma(b_j)} \sum_{n=o}^{\infty} \frac{(-1)^n}{n\,!\,\Gamma(1-\epsilon n+h)} \times$$

$$\times H^{l+q+2,\, n,\, n_1,\, m,\, m_1}_{E+q+2,\,(A+q+2:C),\, F+p+1,\,(B:D)} \left[ \begin{matrix} a \\ b \end{matrix} \middle| \begin{matrix} 1+\lambda,\, (e,\, \theta),\, h-\epsilon n+\theta \\ \lambda+b_q;\, (a,\, \alpha)\,;\, (c,\, \gamma) \\ (f,\, \phi)\,;\, \lambda+a_p \\ (b,\, \beta)\,;\, (d,\, \delta) \end{matrix} \right] \times$$

$$\times \; {}_{p+2}F_{q+2} \left[ \begin{matrix} -n,\, 1+h\,(1-\epsilon)^{-1},\, a_p\,;\, x \\ 1+h-\epsilon\, n,\, h\,(1-\epsilon)^{-1},\, b_q \end{matrix} \right] \tag{2.1}$$

where

$$|\, x-1\, | < 1, \quad |\arg ax\,| < \tfrac{1}{2}\pi\,\zeta_1,$$

$$|\arg bx\,| < \tfrac{1}{2}\pi\,\zeta_2, \quad |\arg a\,| < \tfrac{1}{2}\pi\,\zeta_3,\ |\arg b\,| < \tfrac{1}{2}\pi\,\zeta_4$$

and $h$ and $\epsilon$ are positive integers.

*Proof*; Generalization of Verma's result (1965) is

$$H^{p+s+t,\; 1+v}_{1+l+q,\, p+s+m} \left[ xy \middle| \begin{matrix} 1,\, k_l,\, c_q \\ (a_p,\, A_p),\, (b_s,\, B_s),\, g_m \end{matrix} \right]$$

$$= h \frac{\prod_{j=1}^{p} \Gamma\,(a_j)}{\prod_{j=1}^{q} \Gamma\,(b_j)} \sum_{n=o}^{\infty} \frac{1}{n\,!\;\Gamma\,(1-\epsilon n+h)} \times$$

$$\times\; H^{l+s+u+t,\; l+u}_{l+l+q,\; l+s+u+m} \left[ y \middle| \begin{matrix} 1-n,\, k_l,\, c_q \\ (b_s,\, B_s),\, A_u,\, h-\epsilon n,\, g_m \end{matrix} \right] \times$$

$$\times \; {}_{p+2}F_{u+2} \left[ \begin{matrix} a_p,\, 1+h\,(1-\epsilon)^{-1},\, -n\,;\, \frac{1}{x} \\ A_u,\, h\,(1-\epsilon)^{-1},\, h-\epsilon n+1 \end{matrix} \right] \tag{2.2}$$

where $v < l$, $t < m$, $|\arg xv\,| < \pi$ and $|\arg y\,| < \pi$.

Now multiplying both sides of (2.2) by $y^{\lambda-1}$, integrating it between 0 and $\infty$, (then using the Mellin transform of Fox's $H$-function and also using the elementary relation

$$(\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n}$$

we obtain

$$x^{\lambda} = h \frac{\prod_{j=1}^{p} \Gamma(a_j) \prod_{j=1}^{q} (b_j + B_j \lambda)}{\prod_{j=1}^{p} \Gamma(a_j + A_j \lambda) \prod_{j=1}^{q} \Gamma(b_j)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n (1+\lambda)}{n!\, \Gamma(1+\lambda-n)} \times \frac{\Gamma(h-\epsilon n+\lambda)}{\Gamma(1-\epsilon n+h)} \times$$

$$\times \; {}_{p+2}F_{q+2} \left| \begin{matrix} -n, 1+h(1-\epsilon)^{-1}, a_p ; x \\ 1+h-\epsilon n, h(1-\epsilon)^{-1}, b_q \end{matrix} \right| . \tag{2.3}$$

The expansion (2.1) can easily be established from above by virtue of (1.1).

Results of Jain and Sharma (1968) and Saxena (1971) are the particular cases of (2.1).

*Theorem* 2.2—The expansion formula established here, expreses the generalized function into a product of generalized $H$-function and Fox's $H$-function.

$$x^{\mu} H \left[ \begin{matrix} ax \\ bx \end{matrix} \right] = \sum_{n=0}^{\infty} (2\lambda + 2n)$$

$$\times \; H^{l+r-q+1, n, n_1, m, m_1}_{E+r+2, (A:C), F+p, (B:D)} \left[ \begin{matrix} a \\ b \end{matrix} \middle| \begin{matrix} \mu+\lambda+n, (e, \theta), a_{q+1}+\mu \\ (a, \alpha); (c, \gamma) \\ (f, \phi); a_r+\mu, \mu-\lambda-n \\ (b, \beta); (d, \delta) \end{matrix} \right] \times$$

$$\times \; H^{p, q+1}_{r+2, p} \left[ \frac{1}{x} \middle| \begin{matrix} 1-\lambda-n, (a_r, A_r), 1+\lambda+n \\ (b_p, B_p) \end{matrix} \right] \tag{2.4}$$

where $|\arg ax| < \frac{1}{2}\pi \zeta_1$, $|\arg bx| < \frac{1}{2}\pi \zeta_2$, $|\arg \epsilon| < \pi$,

$|\arg x| < \pi$, $\mu > 0$,

and the series on the r. h. s. converges.

*Proof*: Generalization of a result due to Sharma (*in press*) is

$$x^{\mu} = \frac{\prod_{j=p+1}^{s} \Gamma(b_j - B_j \mu) \prod_{j=p+1}^{r} \Gamma(a_j + A_j \mu)}{\prod_{j=1}^{p} \Gamma(b_j + B_j \mu) \prod_{j=1}^{q} \Gamma(1 - a_j - A_j \mu)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(2\lambda + 2n)\, \Gamma(\lambda + \mu + n)}{\Gamma(\lambda - \mu + n + 1)} \times$$

$$\times H_{r+2,\, s}^{p,\, q+1}\left[\frac{1}{x} \,\middle|\, \begin{matrix} 1-\lambda-n,\, (a_r, A_r),\, 1+\lambda+n \\ (b_s, B_s) \end{matrix}\right]. \tag{2.5}$$

Now on taking $p = s$ in (2.5) and using (1.1) the result is readily obtained.

*Theorem* 2.3—In this theorem we have evaluated a very unique expansion, expressing Fox's $H$-function into a product of two $H$-functions of Fox of different arguments.

$$x^{\mu} H_{r_1+s,\, s_1+r}^{p_1+1,\, q_1+p}\left[ax \,\middle|\, \begin{matrix} (1-\beta_p-\mu),\, (a_{r_1}, A_{r_1}),\, (1-\beta_{p+1}-\mu) \\ (1-\alpha_q-\mu),\, (b_{s_1}, B_{s_1}),\, (1-\alpha_{q+1}-\mu) \end{matrix}\right]$$

$$= \sum_{n=1}^{\infty} (2\lambda + 2n)\, H_{r_1+2,\, s_1}^{p_1,\, q_1+1}\left[a \,\middle|\, \begin{matrix} (1-\mu-\lambda-n),\, (a_{r_1}, A_{r_1}),\, (1-\mu+\lambda+n \\ (b_{s_1}, B_{s_1}) \end{matrix}\right] \times$$

$$\times H_{r+2,\, s}^{p,\, q+1}\left[\frac{1}{x} \,\middle|\, \begin{matrix} (1-\lambda-n),\, (a_r, A_r),\, (1+\lambda+n) \\ (b_s, B_s) \end{matrix}\right] \tag{2.6}$$

where $\mu > 0$, $|\arg ax| < \pi$, $|\arg a| < \pi$, $|\arg x| < \pi$, $\lambda > 0$, and the series on r. h. s. converges.

*Proof*: Let us write (2.5) in the following manner :

$$x^{\mu} = \frac{\prod_{j=p+1}^{s} \Gamma\left[\mu\left(\frac{1-b_j}{\mu} - B_j\right)\right] \prod_{j=q+1}^{r} \Gamma\left[\mu\left(\frac{a_j}{\mu} + A_j\right)\right]}{\prod_{j=1}^{p} \Gamma\left[\mu\left(\frac{b_j}{\mu} + B_j\right)\right] \prod_{j=1}^{q} \Gamma\left[\mu\left(\frac{1-a_j}{\mu} - A_j\right)\right]} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(2\lambda + 2n)\, \Gamma(\lambda + \mu + n)}{\Gamma(\lambda - \mu + n + 1)} \times$$

$$\times H_{r+2,\, s}^{p,\, q+1}\left[\frac{1}{x} \,\middle|\, \begin{matrix} (1-\lambda-n),\, (a_r, A_r)\, (1+\lambda+n) \\ (b_s, B_s) \end{matrix}\right]$$

and now using the multiplication formula and definition of Fox's *H*-function (1961), the expansion formula is readily obtaind.

Results of Abiodun and Sharma (1971) are the special cases of our results.

## ACKNOWLEDGEMENT

Authors are thankful to the referee for the valuable suggestions in the improvement of this paper.

## REFERENCES

Abiodun, R. F. A. and Sharma, B. L. (1971). Summation of series involving generalized hypergeometric function of two variables, *Glasnik. Mat.*, **26**, 253-64.

Erdélyi, A. *et al.* (1953). Higher Transcendental Functions. Vol. I. McGraw-Hill Book Co., Inc., New York.

Fox, C. The *G*-and *H*-functions as symmetrical Fourier kernels. *Trans. Am. math. Soc.*, **98**, 395—429.

Jain, P. C., and Sharma, B. L. (1968). An expansion for generalized function of two variables. *Univ. Nac. Tucuma'n, Rev. Ser. A*, **18**, 7—15.

Munot, P. C. and Kalla, S. L. (1971). On an extension of generalized function of two variables, *Univ, Nac, Tucuma'n, Rev.. Ser. A*. **21**, 67—84.

Saxena, R. K. (1971). Integrals of products of *H*-functions, *Univ. Nac. Tucuma'n, Rev. Ser. A*, **21**, 185—91,

Sharma, B. L. (*in press*). Some expansion formula for the Bessel and *G*-functions, *Bull. Math.*

Verma, A. (1965). A class of expansions of *G*-function and the Laplace transform. *Math. Comp.*, **19**, 664-66.

Verma, R. U. (1971). On the *H*-function of two variables, *An. St. Univ. Al. I. Cuza, Iasi*. T. 17, 17, 103-10.