ON INTEGRALS INVOLVING THE *H*-FUNCTION OF TWO VARIABLES

by S. P. GOYAL, *Department of Mathematics, Banasthali Vidyapith, Banasthali, Rajasthan*

and

S. L. MATHUR, *Department of Mathematics, Government College, Nathdwara, Rajasthan*

In this paper, the authors have established two intergrals involving the product of the *H*-function of one variable and the *H*-function of two variables. The integrals evaluated here are quite general in nature and generalizes almost all the well known integrals due to Gupta and Olkha (1969), Severia (1969), Mittal and Gupta (1972), Gupta (1969), Annadani (1968), Goyal (1970) and others. Many unknown integrals involving the products of various other special functions can be obtained as special cases of present results of the authors. In the course of study, an important and useful particular case of the *H*-function of two variables has also been obtained.

1. INTRODUCTION

a. *The H-function of Two Variables*

The H-function of two variables occurring in this paper has been recently defined and represented by means of the following double Mellin-Barnes type contour integral (Mittal and Gupta 1972, p. 117) :

$$H\left[\begin{matrix}\begin{pmatrix}0, n_1\\ p_1, q_1\end{pmatrix} & \begin{matrix}(a_1 ; \alpha_1, A_1), \ldots, (a_{p_1} ; \alpha_{p_1}, A_{p_1})\\ (b_1 ; \beta_1, B_1), \ldots, (b_{q_1} ; \beta_{q_1}, B_{q_1})\end{matrix} & x\\ \begin{pmatrix}m_2, n_2\\ p_2, q_2\end{pmatrix} & \begin{matrix}(c_1, \gamma_1), \ldots, (c_{p_2}, \gamma_{p_2})\\ (d_1, \delta_1), \ldots, (d_{q_2}, \delta_{q_2})\end{matrix} & \\ \begin{pmatrix}m_3, n_3\\ p_3, q_3\end{pmatrix} & \begin{matrix}(e_1, E_1) \ldots, (e_{p_3}, E_{p_3})\\ (f_1, F_1), \ldots, (f_{q_3}, F_{q_3})\end{matrix} & y\end{matrix}\right] = H[x, y]$$

$$= (1/2\pi i)^2 \int_{L_1}\int_{L_2} \phi(s, t)\ \theta_1(s)\, \theta_2(t)\, x^s\, y^t\ ds\ dt \tag{1.1}$$

where

$$\phi(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)}$$

$$\theta_1(s)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j s)}$$

$$\theta_2(t)=\frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j t)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j t)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j t)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j t)}$$

x and y are not equal to zero and an empty product is interpreted as unity. Also, the non-negative integers n_i, p_i, q_i $(i=1,2,3)$ and m_2, m_3 are such that $0\leqslant n_i\leqslant p_i$, $q_i\geqslant 0$, $0\leqslant m_j\leqslant q_j$ $(i=1,2,3\,;\ j=2,3)$ and all letters $\alpha, \beta, \gamma, \delta, A, B, E, F$ are assumed to be positive.

The contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, that the poles of $\Gamma(d_j-\delta_j s)$ $(j=1,\ldots,m_2)$ lie to the right and those of $\Gamma(1-a_j+\alpha_j s+A_j t)$ $(j=1,\ldots,n)$. $\Gamma(1-c_j+\gamma_j s)$ $(j=1,\ldots,n_2)$ lie to the left of the contour.

The contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j-F_j t)$ $(j=1,\ldots,m_3)$ lie to the right and those of $\Gamma(1-a_j+\alpha_j s+A_j t)$ $(j=1,\ldots,n_1)$, $\Gamma(1-e_j+E_j t)$ $(j=1,\ldots,n_3)$ lie to the left of the contour.

The function defined in (1.1) is an analytic function of x and y (Mittal and Gupta 1972, P. 118) if

$$(i)\quad R=\sum_1^{p_1}(\alpha_j)+\sum_1^{p_2}(\gamma_j)-\sum_1^{q_1}(\beta_j)-\sum_1^{q_2}(\delta_j)<0$$

$$(ii)\quad S=\sum_1^{p_1}(A_j)+\sum_1^{p_3}(E_j)-\sum_1^{q_1}(B_j)-\sum_1^{q_3}(F_j)<0$$

$$(iii)\quad U=\sum_1^{n_1}(\alpha_j)-\sum_{n_1+1}^{p_1}(\alpha_j)-\sum_1^{q_1}(\beta_j)+\sum_1^{m_2}(\delta_j)-\sum_{m_2+1}^{q_2}(\delta_j)+$$

$$+\sum_1^{n_2}(\gamma_j)-\sum_{n_2+1}^{p_2}(\gamma_j)>0$$

$$(iv)\quad V=\sum_1^{n_1}(A_j)-\sum_{n_1+1}^{p_1}(A_j)-\sum_1^{q_1}(B_j)+\sum_1^{m_3}(F_j)-\sum_{m_3+1}^{q_3}(F_j)$$

$$+\sum_1^{n_3}(E_j)-\sum_{n_3+1}^{p_3}(E_j)>0,$$

(*v*) $|\arg x| < (\tfrac{1}{2})\, U\pi, \; |\arg y| < (\tfrac{1}{2})\, V\pi.$

For small values of x and y we have (Mittal and Gupta 1972, p. 119) :

$$\bar{H}[x, y] = o\,(|x|^{\alpha}\,|y|^{\beta})$$

where $\alpha = \min \operatorname{Re}\left(\frac{d_i}{\delta_i}\right) (i = 1, \ldots, m_2)$, $\beta = \min \operatorname{Re}\left(\frac{f_j}{F_j}\right) (j = 1, \ldots, m_3)$

(1.2)

(*b*) ***Symbols Used***

Throughout the present paper, to save space, in the following $\bar{H}$-function of two variables :

$$\bar{H}\left[\begin{array}{c|c|c} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1 \end{pmatrix} & \begin{array}{c} (a_j\,; \alpha_j, A_j)_{1,\, p_1} \\ (b_j\,; \beta_j, B_j)_{1,\, q_1} \end{array} & \begin{array}{c} x \\ y \end{array} \\ \cdots & \cdots & \\ \cdots & \cdots & \end{array}\right]$$

the parameters shown by ... will mean that they are same as that of $\bar{H}[x, y]$ in (1.1)

Also, the following symbols have been used in this paper ;

(*i*) $(a_j;\, \alpha_j, A_j)_{1,p}$ stands for $(a_1;\, \alpha_1, A_1), \ldots, (a_p\,; \alpha_p, A_p)$;

(*ii*) $(a_j;\, \alpha_j, A_j)_{n+1,p}$ stands for $(a_{n+1};\, \alpha_{n+1}, A_{n+1}), \ldots, (a_p\,; \alpha_p, A_p)$;

(*iii*) $(a_j, \alpha_j)_{1,p}$ stands for $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$.

(*c*) ***Results Required.***

The following results will be required in our subsequent analysis :

(*i*) $$H^{m,n}_{p,q}\left[ax^{\sigma} \,\middle|\, \begin{array}{l} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,p} \end{array}\right]$$

$$= \sum_{h=1}^{m} \sum_{r=0}^{\infty} \frac{(-1)^r \prod\limits_{\substack{j=1 \\ j \neq h}}^{m} \Gamma(b_j - \beta_j\, \rho_r) \prod\limits_{1}^{n} \Gamma(1 - a_j + \alpha_j + \rho_r)\, a^{\rho_r}\, x^{\sigma \rho_r}}{r!\, \prod\limits_{m+1}^{q} \Gamma(1 - b_j + \beta_j\, \rho_r) \prod\limits_{n+1}^{p} \Gamma(a_j - \alpha_j\, \rho_r)\, \beta_h} \tag{1.3}$$

where $\beta_h(b_j + \lambda) \neq \beta_j(b_h + r)$ $j \neq h$; $j, h = 1, \cdots, m$; $\lambda, r = 0, 1, 2, \cdots$

$\rho_r = \dfrac{b_h + r}{\beta_h}$ and provided that either

(*i*) $\sigma > 0$, $\delta = \sum\limits_{1}^{q} \beta_j - \sum\limits_{1}^{p} \alpha_j > 0$, $ax^{\sigma} \neq 0$ or (*ii*) $\sigma > 0, \quad \delta = 0, \quad 0 < | ax^{\sigma} | < \eta^{-1}$

where $\eta = \prod_{1}^{p} (\alpha_j)^{\alpha_j} \prod_{1}^{q} (\beta_j)^{-\beta_j}$

The result (1.5) was given by [Braaksma (1963), p. 278].

(*ii*) $\int_0^t x^{\rho-1} (t-x)^{\sigma-1} H [yx^{\lambda} (t-x)^{\mu}, zx^{\delta} (t-x)^{\nu}] \, dx$

$$= t^{\rho+\sigma-1} H \left[\begin{pmatrix} 0, n_1+2 \\ p_1+2, q_1+1 \end{pmatrix} \left| \begin{matrix} (1-\rho ; \lambda, \delta), (1-\sigma ; \mu, \nu), (a_j ; \alpha_j, A_j)_{1,p_1} \\ (b_j ; \beta_j, B_j)_{1,q_1}, (1-\rho-\sigma ; \lambda+\mu, \delta+\nu) \\ \cdots \\ \cdots \end{matrix} \right| \begin{matrix} yt^{\lambda+\mu} \\ zt^{\delta+\nu} \end{matrix} \right] \tag{1.4}$$

where $H [yx^{\lambda} (t-x)^{\mu}, zx^{\delta} (t-x)^{\nu}]$ is the H-function of two variables as defined in (1.1)

The integral (1.4) is valid under the following set of conditions ·

(*i*) $\lambda > 0$, $\mu > 0$, $\delta > 0$, $\nu > 0$, Re $(\rho+\lambda\alpha+\delta\beta) > 0$, Re $(\sigma+\mu\alpha+\nu\beta) > 0$ where α, β stand for the quantities mentioned in (1.2).

(*ii*) The conditions (*i*) to (*v*) given in section 1 are satisfied.

The integral (1.4) can be easily proved by first expressing the H-function of two variables occurring in the left-hand side of (1.4) in terms of Mellin-Barnes type contour integral with the help of (1.1), then change the order of integrations and evaluate the x-integral thus obtained with the help of the definition of Beta function, we get the right-hand side of integral (1.4) using (1.1) in it.

$$\operatorname*{Lt}_{y \to 0} H \left[\begin{matrix} \begin{pmatrix} 0, p_1 \\ p_1, q_1 \end{pmatrix} \\ \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} \\ \begin{pmatrix} 1, p_3 \\ p_3, q_3+1 \end{pmatrix} \end{matrix} \left| \begin{matrix} (a_j ; \alpha_j, A_j)_{1,p_1} \\ (b_j ; \beta_j, B_j)_{1,q_1} \\ (c_j, \gamma_j)_{1,p_2} \\ (d_j, \delta_j)_{1,q_2} \\ (e_j, E_j)_{1,p_3} \\ (0, 1)(f_j, F_j)_{1,q_3} \end{matrix} \right| \begin{matrix} x \\ \\ y \end{matrix} \right]$$

$$= \frac{\prod_{1}^{p_3} \Gamma(1-e_j)}{\prod_{1}^{q_3} \Gamma(1-f_j)} H^{m_2, p_1+n_2}_{p_1+p_2, q_1+q_2} \left[x \left| \begin{matrix} (a_j, \alpha_j)_{1,p_1}, (c_j, \gamma_j)_{1,p_2} \\ (d_j, \delta_j)_{1,q_2}, (b_j, \beta_j)_{1,q_1} \end{matrix} \right. \right] \tag{1.5}$$

Proof of (1.5)—On using (1.1) and a known result (Gupta and Jain 1966, p. 594) of (1.6), it reduces to the following expression :

$$(a)\quad \operatorname*{Lt}_{y\to 0}\ (1/2\pi i)\int_{L_1}\frac{\prod_{1}^{n_2}\Gamma(1-c_j+\gamma_j s)\prod_{1}^{m_2}\Gamma(d_j-\delta_j s)\,x^s}{\prod_{n_2+1}^{p_2}\Gamma(c_j-\gamma_j s)\prod_{m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)}$$

$$\times H^{1,p_1+p_3}_{p_1+p_3,q_1+q_3+1}\left[y\,\middle|\,\begin{matrix}(a_j-\alpha_j s, A_j)_{1,p_1},(e_j,E_j)_{1,p_3}\\(0,1),(b_j-\beta_j s,B_j)_{1,q_1},(f_j,F_j)_{1,q_3}\end{matrix}\right]ds.$$

Now, on using the known results [Gupta and Jain 1966, p. 601, (4.9) ; Wright (1935), p. 287] in the expression (a), we get

$$(b)\quad (1/2\pi i)\int_{L_1}\frac{\prod_{1}^{n_2}\Gamma(1-c_j+\gamma_j s)\prod_{1}^{m_2}\Gamma(d_j-\delta_j s)\,x^s}{\prod_{n_2+1}^{p_2}\Gamma(c_j-\gamma_j s)\prod_{m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)}$$

$$\times\left\{\operatorname*{Lt}_{y\to 0}\sum_{r=0}^{\infty}\frac{\prod_{1}^{p_1}\Gamma(1-a_j+\alpha_j s+A_j r)\prod_{1}^{p_3}\Gamma(1-e_j+E_j r)\,(-y)^r}{\prod_{1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j r)\prod_{1}^{q_3}\Gamma(1-f_j+F_j r)\,r\,!}\right\}ds.$$

On taking the limits in the expression (b) and then using the result (Gupta and Jain 1966, p. 594) in it, we get the right hand side of (1.5).

2. Main Integrals

The following integrals have been evaluated in this paper :

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}H^{m,n}_{p,q}\left[ax^u\,\middle|\,\begin{matrix}(g_j,G_j)_{1,p}\\(h_j,H_j)_{1,q}\end{matrix}\right]$$

$$\times H\,[yx^\lambda(t-x)^\mu, zx^\delta(t-x)]\,dx$$

$$=t^{\rho+\sigma-1}\sum_{h=1}^{m}\sum_{r=0}^{\infty}\frac{(-1)^r\prod_{j=1,\,j\neq h}^{m}\Gamma(h_j-H_j\rho_r)\prod_{1}^{n}\Gamma(1-g_j+G_j\rho_r)\,t^{u\rho_r}\,a^{\rho_r}}{\prod_{m+1}^{q}\Gamma(1-h_j+H_j\rho_r)\prod_{n+1}^{p}\Gamma(g_j-G_j\rho_r)\,r\,!\,H_h}$$

$$\times H\left[\begin{pmatrix}0,\ n_1+2\\ p_1+2,\ q_1+1\\ \cdots\\ \cdots\end{pmatrix}\,\middle|\,\begin{matrix}(1-\rho-u\rho_r;\lambda,\delta),(1-\sigma;\mu,\nu),(a_j;\alpha_j,A_j)_{1,p_1}\\(b_j;\beta_j\,B_j)_{1,q_1},(1-\rho-\sigma-u\rho_r;\lambda+\mu,\delta+\nu)\\ \cdots\\ \cdots\end{matrix}\,\middle|\,\begin{matrix}yt^{\lambda+\mu}\\ zt^{\delta+\nu}\end{matrix}\right]\qquad(2.1)$$

where (i) $\rho_r = \frac{h_h+r}{H_h}$ (ii) $H\ [yx^\lambda\ (t-x)^\mu.\ zx^\delta\ (t-x)^\nu]$ stands for the H-function of two variables as defined in (1.1).

The integral (2.1) is valid under the following sets of conditions :

(i) $u > 0,\ B = \sum_1^n (G_j) - \sum_{n+1}^p (G_j) + \sum_1^m (H_j) - \sum_{m+1}^q (H_j) > 0$

$|\arg a| < (\tfrac{1}{2}) B\pi,\ B' = \sum_1^q (H_j) - \sum_1^p (G_j) > 0,$

(ii) $\lambda > 0,\ \mu > 0,\ \delta > 0,\ \nu > 0$. $\mathrm{Re}\,(\sigma+\mu\alpha+\nu\beta) > 0,$

$\mathrm{Re}\,(\rho+u\frac{h_i}{H_i} + \lambda\alpha + \delta\beta) > 0,\ (i = 1, \ldots, m)$, where α and β are the quantities as mentioned in (1.2).

(iii) The conditions (i) to (v) given in section 1 are satisfied.

$$\int_0^\infty x^{\rho-1} H_{p,q}^{m,n}\left[ax^u \middle| \begin{matrix}(g_j, G_j)_{1,p} \\ h_0, H_0), (h_j, H_j)_{1,q}\end{matrix}\right]$$

$$\times H_{f,l}^{k,0}\left[bx \middle| \begin{matrix}(r_j, R_j)_{1,f} \\ (u_j, U_j)_{1,l}\end{matrix}\right] H\ [yx^\lambda, zx^\delta]\ dx$$

$$= \frac{1}{b^\rho} \sum_{h=1}^m \sum_{r=0}^\infty \frac{(-1)^r \prod_{\substack{j=1 \\ j\neq h}}^m \Gamma(h_j - H_j\rho_r) \prod^n \Gamma(1-g_j+G_j\rho_r)\, a^{\rho_r}}{r! \prod_{m+1}^q \Gamma(1-h_j+H_j\rho_r) \prod_{n+1}^p \Gamma(g_j - G_j\rho_r)\, b^{u\rho_r}}$$

$$\times H\left[\begin{pmatrix} 0, n_1+k \\ p_1+1, q_1+f \end{pmatrix} \middle| \begin{matrix} M \\ N \end{matrix} \middle| \begin{matrix} \frac{y}{b^\lambda} \\ \frac{z}{b^\delta} \end{matrix}\right] \begin{matrix} \ldots \\ \ldots \end{matrix} \quad (2.2)$$

where (i) $\rho_r = \frac{h_h+r}{H_h}$ (ii) $H\ [yx^\lambda, zx^\delta]$ stands for the H-function of two variables as defined in (1.1).

(iii) M stands for $(a_j\,;\alpha_j, A_j)_{1,n_1}, (1-u_j-(\rho+u\rho_r)U_j\,;\lambda U_j, \delta U_j)_{1,l},$
$(a_j\,;\alpha_j, A_j)_{n_1+1,p_1}.$

(iv) N stands for $(b_j\,;\beta_j, B_j)_{1,q_1}, (1-r_j-(\rho+u\rho_r)R_j\,;\lambda R_j, \delta R_j)_{1,f}.$

The integral (2.2) is valid under the following sets of conditions :

(*i*) Set of conditions (*i*) given just above is satisfied

(*ii*) $C = \sum_{1}^{k} (U_j) - \sum_{k+1}^{l} (U_j) - \sum_{1}^{f} (R_j) > 0, |\arg b| < (\frac{1}{2}) C \pi.$

(*iii*) $\lambda > 0, \quad \delta > 0, \quad \text{Re}\,(\rho + u\,(h_i/H_i) + (u_j/U_j) + \lambda\alpha + \delta\beta) > 0 \quad (i=1, \cdots, m\,;\, j=1, \ldots, k)$, ($\alpha$ and β stand for the quantities as mentioned in (1.2),

(*iv*) The conditions (*i*) to (*v*) given in section 1 are satisfied.

Proofs of (2.1) and (2.2)

Proof of (2.1)—On using the series expansion given by (1.3) in the left hand side of (2.1), changing the order of integration and summation) [which is easily seen to be justified under the conditions mentioned with (2.1)] in the result thus obtained, we get

$$\sum_{h=1}^{m} \sum_{r=0}^{\infty} \frac{(-1)^r \prod_{\substack{j=1 \\ j\neq h}}^{m} \Gamma(h_j - H_j\,\rho_r) \prod_{1}^{n} \Gamma(1 - g_j + G_j\,\rho_r)\, a^{\rho_r}}{r!\, \prod_{m+1}^{q} \Gamma(1 - h_j + H_j\,\rho_r) \prod_{n+1}^{p} \Gamma(g_j - G_j\,\rho_r)\, H_h}$$

$$\times \int_0^t x^{\rho + u\rho_r - 1}\,(t-x)^{\sigma-1}\, H\,[yx^{\lambda}\,(t-x)^{\mu},\, zx^{\delta}\,(t-x)^{\nu}]\, dx. \cdots (a)$$

Now, on evaluating the x-integral in the expression (*a*) with the help of (1.4), we get the right-hand side of (2.1).

Proof of (2.2) — On writing the series expansion for $H_{p,q}^{m,n}\,[ax^u]$ in the left hand side of (2.2) with the help of (1.3), changing the order of integration and summation and finally evaluating the resulting x-integral with the help of the integral due to Mittal and Gupta [1972. P. 121] in the following form :

$$\int_0^{\infty} x^{\rho-1}\, H_{f,l}^{k,0}\left[bx \,\middle|\, \begin{matrix} (r_j, R_j)_{1,f} \\ (u_j, U_j)_{1,s} \end{matrix} \right] H\,[yx^{\lambda}, zx^{\delta}]\, dx$$

$$= b^{-\rho} H \left[\begin{matrix} \begin{pmatrix} 0, & n_1 + k \\ p_1 + l, & q_1 + f \end{pmatrix} \\ \cdots \\ \cdots \end{matrix} \left| \begin{matrix} (a_j\,;\, \alpha_j, A_j)_{1,n_1}, (1 - u_j - \rho U_j\,;\, \lambda U_j, \delta U_j)_{1,l}, \\ (a_j\,;\, \alpha_j, A_j)_{n_1+1, p_1} \\ (b_j\,;\, \beta_j, B_j)_{1,q_1}, (1 - r_j - \rho R_j\,;\, \lambda R_j, \delta R_j)_{1,f} \\ \cdots \\ \cdots \end{matrix} \right| \begin{matrix} yb^{-\lambda} \\ zb^{-\delta} \end{matrix} \right]$$

(2.3)

(where $C > 0$, $|\arg b| < (\frac{1}{2}) C\pi$, $\lambda > 0$, $\delta > 0$, and $\mathrm{Re}(\rho+(u_j/U_j)+\lambda\alpha+\delta\beta) > 0$ $(j = 1,\ldots, k)$, [where α and β stand for the quantities as mentioned in (1.2)] and the conditions (*i*) to (*v*) given in section 1 are satisfied), we arrive at the required result (2.2).

3. Particular Cases

Since the H-function of one variable and the H-function of two variables involved in (2.1) and (2.2) are very general in nature and include almost all the special functions studied so far as mentioned by Gupta and Jain (1966), Mittal and Gupta (1972) and Goyal (1975) we can obtain alarge number of integrals involving the products of various special functions, but we record here only few of them due to lack of space.

(*i*) If we take take $n_1 = p_1 = q_1 = 0$ in (2.1) and (2.2) and make use of a known result (Mittal and Gupta 1972, p. 119), we get the following integrals involving the product of H-functions :

$$(a)\quad \int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[ax^u \,\middle|\, \begin{matrix}(g_j, G_j)_{1,p}\\ (h_j, H_j)_{1,q}\end{matrix}\right]$$

$$\times H_{p_2,q_2}^{m_2,n_2}\left[yx^\lambda (t-x)^\mu \,\middle|\, \begin{matrix}(c_j, \gamma_j)_{1,p_2}\\ (d_j, \delta_j)_{j,q_2}\end{matrix}\right]$$

$$\times H_{p_3,q_3}^{m_3,n_3}\left[zx^\delta (t-x)^\nu \,\middle|\, \begin{matrix}(e_j, E_j)_{1,p_3}\\ (f_j, F_j)_{1,q_3}\end{matrix}\right] dx$$

$$= t^{\rho+\sigma-1}\sum_{h=1}^{m}\sum_{r=0}^{\infty} \frac{(-1)^r \prod_{j=1, j\neq h}^{m}\Gamma(h_j - H_j\rho_r)\prod_{1}^{n}\Gamma(1-g_j+G_j\rho_r)\, a^{\rho_r}\, t^{u\rho_r}}{r!\prod_{m+1}^{q}\Gamma(1-h_j+H_j\rho_r)\prod_{n+1}^{p}\Gamma(g_j-G_j\rho_r)\, H_h}$$

$$\times H\left[\begin{matrix}\begin{pmatrix}0, 2\\ 2, 1\end{pmatrix}\\ \cdots\\ \cdots\end{matrix}\,\middle|\,\begin{matrix}(1-\rho-u\rho_r\,;\lambda,\delta),(1-\sigma\,;\mu,\nu)\\ (1-\rho-\sigma-u\rho_r\,;\lambda+\mu,\delta+\nu)\\ \cdots\\ \cdots\end{matrix}\,\middle|\,\begin{matrix}yt^{\lambda+\mu}\\ zt^{\delta+\nu}\end{matrix}\right] \tag{3.1}$$

where $\rho_r = \dfrac{h_h+r}{H_h}$ and the conditions easily obtainable from (2.1) are satisfied.

$$(b) \int_0^\infty x^{\rho-1} H_{p,q}^{m,n}\left[ax^u \middle| \begin{matrix}(g_j, G_j)_{1,p}\\(h_j, H_j)_{1,q}\end{matrix}\right] \times H_{f,l}^{k,0}\left[bx \middle| \begin{matrix}(r_j, R_j)_{1,f}\\(u_j, U_j)_{1,l}\end{matrix}\right]$$

$$\times H_{p_2,q_2}^{m_2,n_2}\left[yx^\lambda \middle| \begin{matrix}(c_j, \gamma_j)_{1,p_2}\\(d_j, \delta_j)_{1,q_2}\end{matrix}\right] H_{p_3,q_3}^{m_3,n_3}\left[zx^\delta \middle| \begin{matrix}(e_j, E_j)_{1,p_3}\\(f_j, F_j)_{1,q_3}\end{matrix}\right] dx$$

$$= \frac{1}{b^\rho} \sum_{h=1}^{m} \sum_{r=0}^{\infty} \frac{(-1)^r \prod_{j=1, j\neq h}^{m} \Gamma(h_j - H_j \rho_r) \prod_1^n \Gamma(1-g_j+G_j \rho_r)\, a^{\rho_r}}{r! \prod_{m+1}^{q} \Gamma(1-h_j+H_j \rho_r) \prod_{n+1}^{q} \Gamma(g_j - G_j \rho_r)\, b^{u\rho_r} H_h}$$

$$\times H\left[\begin{pmatrix}0, k\\ l, f\end{pmatrix} \begin{matrix} \\ \\ \cdots \\ \cdots\end{matrix} \middle| \begin{matrix}(1-u_j-(\rho+u\rho_r)U_j ;\ \lambda U_j, \delta U_j)_{1,l}\\(1-r_j-(\rho+u\rho_r)R_j ;\ \lambda R_j, \delta R_j)_{1,f}.\\ \cdots \\ \cdots\end{matrix} \middle| \begin{matrix}\dfrac{y}{b^\lambda}\\ \dfrac{z}{b^\delta}\end{matrix}\right] \tag{3.2}$$

where $\rho_r = \dfrac{h_h+r}{H_h}$ and the conditions easily obtainable from the conditions given with (2.2) are satisfied.

(*ii*) If we put $m_3 = 1$, $n_1 = p_1$, $n_3 = p_3$, $f_1 = 0$, $F_1 = 1$ in (2.1) and (2.2) replace q_3 by q_3+1, f_{j+1} by f_j $(j = 1,\ldots, q)$ therein, after making use of (1.5) and slight changes in the parameters, we get the following interesting integrals :

$$(c) \int_0^t x^{\rho-1}(t-x)^{\sigma-1} H_{p,q}^{m,n}\left[ax^u \middle| \begin{matrix}(g_j, G_j)_{1,p}\\(h_j, H_j)_{1,q}\end{matrix}\right] \times H_{p_1,q_1}^{m_1,n_1}\left[yx^\lambda(t-x)^\mu \middle| \begin{matrix}(a_j, \alpha_j)_{1,p_1}\\(b_j, \beta_j)_{1,q_1}\end{matrix}\right] dx$$

$$= t^{\rho+\sigma-1} \sum_{h=1}^{m} \sum_{r=0}^{\infty} \frac{(-1)^r \prod_{j=1, j\neq h}^{m} \Gamma(h_j - H_j \rho_r) \prod_1^n \Gamma(1-g_j+G_j \rho_r)\, a^{\rho_r}}{r! \prod_{m+1}^{q} \Gamma(1-h_j+H_j \rho_r) \prod_{n+1}^{p} \Gamma(g_j - G_j \rho_r)\, H_h}$$

$$\times H_{p_1+2,q_1+1}^{m_1,n_1+2}\left[yt^{\lambda+\mu} \middle| \begin{matrix}(1-\rho-u\rho_r,\lambda), (1-\sigma, \mu), (a_j, \alpha_j)_{1,p_1}\\(b_j, \beta_j)_{1,q_1}, (1-\rho-\sigma-u\rho_r, \lambda+\mu)\end{matrix}\right] \tag{3.3}$$

where $\rho_r = \dfrac{h_h+r}{H_h}$ and the integral is valid under the following set of conditions :

(*i*) The set of conditions (*i*) given in (2.1) is satisfied ;

(*ii*) $\lambda > 0$, $\mu > 0$ $\text{Re}\,(\rho+u\,(h_i/H_i)+\lambda\,(b_j/\beta_j)) > 0$ $(i=1,\ldots,m\,;\ j=1,\ldots,m_1)$ and $\text{Re}\,(\sigma+\mu\,(b_j/\beta_j)) > 0$ $(j=1,\ldots,m_1)$;

(*iii*) $D = \sum_{1}^{n_1}(\alpha_j) - \sum_{n_1+1}^{p_1}(\alpha_j) + \sum_{1}^{m_1}(\beta_j) - \sum_{m_1+1}^{q_1}(\beta_j) > 0,$

$|\arg y| < (\tfrac{1}{2})\,D\,\pi$

and

(*d*)

$$\int_0^\infty x^{\rho-1}\, H_{p,q}^{m,n}\left[ax^u \,\middle|\, \begin{matrix}(g_j, G_j)_{1,p}\\ (h_j, H_j)_{1,q}\end{matrix}\right] \times H_{f,v}^{k,0}\left[bx \,\middle|\, \begin{matrix}(r_j, R_j)_{1,f}\\ (u_j, U_j)_{1,l}\end{matrix}\right] H_{p_1,q_1}^{m_1,n_1}\left[yx^\lambda \,\middle|\, \begin{matrix}(a_j, \alpha_j)_{1,p_1}\\ (b_j, \beta_j)_{1,q_1}\end{matrix}\right] dx$$

$$= \frac{1}{b^\rho}\sum_{h=1}^{m}\sum_{r=0}^{\infty} \frac{(-1)^r \prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(h_j-H_j\,\rho_r)\prod_{1}^{n}\Gamma(1-g_j+G_j\,\rho_r)\,a^{\rho_r}}{r\,!\prod_{m+1}^{q}\Gamma(1-h_j+H_j\,\rho_r)\prod_{n+1}^{p}\Gamma(g_j-G_j\,\rho_r)\,b^{\rho u_r}\,H_h}$$

$$\times H_{p_1+l,\ q_1+f}^{m_1,\ n_1+k}\left[yb^{-\lambda} \,\middle|\, \begin{matrix}(a_j, \alpha_j)_{1,n_1}, (1-u_j-(\rho+u\,\rho_r)\,U_j, \lambda\,U_j)_{1,l}, (a_j, \alpha_j)_{n_1+1,p_1}\\ (b_j, \beta_j)_{1,q_1}, (1-r_j-(\rho+u\,\rho_r)\,R_j, \lambda R_j)_{1,f}\end{matrix}\right] \tag{3.4}$$

where $\rho_r = \dfrac{h_h+r}{H_h}$ and the integral is valid under the following set of conditions :

(*i*) The set of conditions (*i*) given with (2.1) is satisfied.

(*ii*) The set of conditions (*iii*) given with (3.3) is satisfied.

(*iii*) $\lambda > 0$, $\text{Re}\,(\rho+u\,(h_i/H_i)+(u_j/U_j)+\lambda\,(b_w/\beta_w)) > 0$ $(i = 1, \ldots\ldots, m\,;$ $j = 1, \ldots, k\,;\ w = 1, \ldots, m_1)$.

The result is the generalization of the integrals obtained earlier by Annadani (1968).

If we take $m=1$, $n=p$, $h_1=0$, $G_i = H_j = 1$, $(i = 1, ..., p\ ;\ j = 1, ..., q)$ $t = 1$ in (3.3) and use the known results [Gupta and Jain 1966, p. 598, (4.1) ; Erdélyi 1953, p. 208, (1)] in it, we get an integral obtained earlier by Gupta and Olkha (1969, p. 207).

Also, if we take $m=1$, $n=p$, $\mu = \nu = 0$, $h_1=0$, $G_i=H_j=1$ $(i = 1, ..., p\ ;$ $j = 1, ..., q)$ in (2.1), put all α's, β's, γ's, δ's, A's, B's, E's, F's equal to one in it and make use of the known results [Gupta and Jain 1966 p. 598, (4.1) ; Erdélyi, 1953, p. 208, (1)] therein then we get a result recently obtained by the author Goyal 1970, p. 222) which is the generalization of the result due to Severia (1968, p. 21).

Again, if we take $n_1 = p = n = 0, m= q = 1$, $h_1 = 0$, $H_1 = 1$, $u = 1$, in (2.2) and let $a \rightarrow 0$ therein, we get another known result due to Mittal and Gupta (1972, 121).

Lastly, if we get $m = q = 1$, $n_1=n = p = 0$, $h_1 = 0$, $u = H_1 = 1$ in (3.2), put all R's, Us', γ's, δ's, E's, F's equal to unity in it and let $a \rightarrow 0$, therein, we get another integral involving the product of G-functions obtained by Gupta (1969, p. 193).

ACKNOWLEDGEMENT

The authors are grateful to Dr. K. C. Gupta of M. R. Engineering College, Jaipur, for his kind help and useful suggestions made during the preparation of this paper.

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