AN INVERSION FORMULA FOR AN INTEGRAL TRANSFORM

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The object of the present paper is to investigate an inversion formula for an integral transform, whose kernel has a Mellin-Barnes type integral representation, by the application of fractional integration operators due to Kalla and Saxena (1969), which are capable of cancelling more gamma functions from the integrand comparative to Erdélyi-Kober or $L-L^{-1}$ operators. On account of most general character of the kernel various results given earlier by many authors, notably Fox (1963) and Saxena (1966), can be particularized from the result given in this paper,

INTRODUCTION

Fox (1963) and Saxena (1966) has shown that with the help of fractional integration operators due to Erdélyi (1940) and Kober (1940), it is possible to establish the reciprocities which resemble

$$g(x) = \int_0^\infty k(xu) f(u)\, du \tag{1.1}$$

$$f(x) = \int_0^\infty k(xu) g(u)\, du \tag{1.2}$$

usually known as the generalized Fourier integral transform, closely although the functional equation satisfied by the kernel is much more general than

$$K(s) K(1-s) = 1 \tag{1.3}$$

where $K(s)$ denotes the Mellin transform of the kernel $k(x)$.

Let

$$R_{p,q,r}(x) = \frac{1}{2\pi i} \int_c \chi(s)\, x^{-s}\, ds \tag{1.4}$$

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where

$$\chi(s) = \prod_{i=1}^{p} \Gamma\left(\frac{a_j+s}{\zeta_i}\right) \prod_{i=1}^{q} \frac{\Gamma\left(\frac{d_i-s}{\mu_i}\right)\Gamma\left(\frac{c_i+s}{\mu_i}\right)}{\Gamma\left(\frac{b_i-s}{\mu_i}\right)\Gamma\left(\frac{c_i+s}{\mu_i}+m_i\right)}$$

$$\times \prod_{i=1}^{r} \frac{\Gamma\left(\frac{e_i+s}{\lambda_i}\right)\Gamma\left(\frac{f_i-s}{\lambda_i}\right)}{\Gamma\left(\frac{h_i+s}{\lambda_i}\right)\Gamma\left(\frac{f_i-s}{\lambda_i}+n_i\right)} \tag{1.5}$$

From (1.4) we infer that the Mellin transform of $R_{p,q,r}(x)$ is

$$\chi(x). \tag{1.6}$$

We now define an integral transform over the interval $(0, \infty)$ by the integral equation

$$\phi(x) = \int_0^\infty R_{p,q,r}(xu)\, h(u)\, du. \tag{1.7}$$

Yet we can find an inversion formula for the abeve integral transform (1.7) by the method of L and L^{-1} operators, a method recently given by Fox (1971) or by the application of Erdélyi-Kober operators used earlier by Fox (1963) and Saxena (1966) but the application of L and L^{-1} or Erdélyi-Kober operators eliminates only one or two gamma functions from the integrand respectively, whereas Kalla and Saxena's (1969) operators are capable of eliminating four gamma functions, two each from the numerator and the denominator and hence Kalla and Saxena's operators are more suitable to find the inversion formula for the above integral transform.

The integral transform (1.7) exist under the following set of conditions :

(*i*) $h(x)$ belongs to $L_2(0, \infty)$.

(*ii*) $x > 0$.

(*iii*) p is *a* positive integer and q and r are non-negative integers.

(*iv*) μ_i and $\lambda_j > 0$ for $i = 1, 2, \ldots, q$, $j = 1\ 2, \ldots, r$.

(*v*) m_i, $n_j = 0, 1, 2, \ldots$; for $1 = 1, 2 \ldots, q$; $j = 1, 2, \ldots, r$.

(*vi*) All the poles of thc integrand in (1.4) are simple.

The contour C is such that all the poles of $\Gamma\left(\frac{a_i+s}{\zeta_i}\right)$, $\Gamma\left(\frac{c_i+s}{\mu_i}\right)$, $\Gamma\left(\frac{e_i+s}{\lambda_i}\right)$ lie to the left and those of $\Gamma\left(\frac{d_i-s}{\mu_i}\right)$ and $\Gamma\left(\frac{f_i-s}{\lambda_i}\right)$ to the right of it.

(*vii*) The contour C is a straight line parallel to the imaginary axis in the complex s-plane given by $s = \frac{1}{2} + it$, t being real and $-\infty < t < \infty$.

(*viii*) $a_i \neq a_j$, $i \neq j$, $i = 1, 2, \ldots, p$. Similar conditions hold for other.

The function $R_{p,q,r}$ will also be written as

$$R_{p,q,r}\left(x \,\middle|\, \begin{matrix} c_q + m_q, h_r, d_q, h_r \\ a_p, e_q, e_r, b_q, f_r + n_r \end{matrix} \right) \tag{1.8}$$

whereever necessary for definiteness.

2. Mellin Transform

The Mellin transform of $h(u)$ will be represented by $M\{h(u)\}$. If $M\{h(u)\} = H(s)$ we shall also write $h(u) = M^{-1}\{H(S)\}$ and M^{-1} will indicate the inverse Mellin transform.

Formally, we have

$$H(s) = M\{h(u)\} = \int_0^\infty h(u)\, u^{s-1}\, du \tag{2.1}$$

and

$$h(u) = M^{-1}\{H(u)\} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} H(s)\, u^{-s} ds \tag{2.2}$$

We shall make use of L_2-space theory in which the conditions are simple.

If

$$h(u) \in L_2(0, \infty)$$

and lim is with index 2 then

$$H(s) = M\{h(z)\} = \lim_{N \to \infty} \int_{1/N}^{N} h(z)\, z^{s-1}\, dz \tag{2.3}$$

and also

$$H(s) \in L_2\left(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty\right) \tag{2.4}$$

If

$$H(s) \in L_2\left(\tfrac{1}{2} - i\infty, \tfrac{1}{2} + i\infty\right)$$

then

$$h(z) = M^{-1}\{H(s)\} = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{1/2 - iN}^{1/2 + iN} H(s)\, z^{-s} ds \tag{2.5}$$

where $h(z) \in L_2(0, \infty)$ (Saxena 1966, Theorem 71, p. 94).

The following lemmas recently proved by Fox (1971, 1975) will also be required in the proof.

Lemma 1—(*i*) $x > 0$, (*ii*) $h(z)$ and $g(z)$ both $\in L_2(0, \infty)$, (*iii*) $M\{h(z)\} = H(s)$, $M\{g(z)\} = G(s)$, and $G(s)$ is bounded on the line $s = \frac{1}{2} + it$, $-\infty < t < \infty$, then

$$\int_0^\infty g(xz)\, h(z)\, dz \in L_2(0\ \infty) \tag{2.6}$$

and

$$M\left\{\int_0^\infty g(xz)\, h(z)\, dz\right\} = G(s)\, H(1-s) \tag{2.7}$$

where the integrals of (2. 6) and (2.7) are regarded as functions of x.

Lemma 2—If (*i*) $\alpha > 0$, $\frac{1}{2}\alpha + \beta > 0$, $u > 0$, (*ii*) $s = \sigma + i\rho$, σ and ρ both real : $H(s) \in L(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$ then

$$L^{-1}\left\{\frac{1}{2\pi i}\int_C \Gamma(\alpha s + \beta)\, H(s)\, u^{-\alpha s - \beta}\, ds\right\}$$

$$= \frac{1}{2\pi i}\int_C H(s)\, x^{\alpha s + \beta - 1}\, ds \tag{2.8}$$

where, for both integrals, the contour C may be the line $\sigma = \frac{1}{2}$.

Lemma—3 If (*i*) $x > 0$, (*ii*) $a_j \geqslant 0$, $j = 1, 2, .., p$, (*iii*) $b_j > 0$, $j = 1, 2, \ldots, q$ (*iv*) $h(x) \in L_2(0, \infty)$ then

$$\int_0^\infty R_{p,q,r}(xu)\, h(u)\, du \in L_2(0, \infty) \tag{2.9}$$

and

$$M\left\{\int_0^\infty R_{p,q,r}(xu)\, h(u)\, du\right\} \chi(s)\, H(1-s) \tag{2.10}$$

where the left-hand side of (2.9) is regarded as a function of x.

The proof of Lemma 3 is similar to that of Saxena (1966).

3. Fractional Integration Operators

Various definitions of fractional integrations operators have been given from time to time by many authors including Kober (1940), Erdélyi (1940), Weyl (1917), Saxena (1967) and Kalla and Saxena (1969). Kalla and Saxena's definitions of fractional integration are slight variance of Saxena's definition.

Here we use the definitions given by Saxena and generalized by Kalla and Saxena. The operators will be denoted by $I[f(x)]$ and $R[f(x)]$ where, in both cases, f (x) is fractionally integrated. The definitions are as follows :

$$1 \leqslant p, q < \infty, p^{-1} + q^{-1} = 1, \gamma > 0;$$

$$\text{Re}(\alpha) > 0, \text{Re}(\eta) > -\frac{1}{q}, \text{Re}(\delta) > -\frac{1}{p};$$

$$\text{Re}(1-\alpha) > m, m = 0, 1, 2, \ldots; \beta \neq 0, -1, -2, \ldots;$$

$$f(x) \in L_p(0, \infty), \tag{3.1}$$

$$I[f(x)] = [\alpha, \beta, m, \gamma, \eta : f(x)]$$

$$= \frac{\gamma\, x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x F\left(\alpha, \beta+m; \beta; \frac{t^\gamma}{x^\gamma}\right) t^\eta f(t)\, dt \tag{3.2}$$

$$R[f(x)] = R[\alpha, \beta, m, \gamma, \delta : f(x)]$$

$$= \frac{\gamma\, x^{\delta}}{\Gamma(1-\alpha)} \int_x^\infty F\left(\alpha, \beta+m; \beta\; \frac{x^\gamma}{t^\gamma}\right) t^{-\delta-1} f(t)\, dt \tag{3.3}$$

where $F(a, b; c; x)$ denotes the Gauss's hypergeometric function, $\alpha, \beta. \eta$, and δ are complex parameters.

Condition (3.1) ensures that $I[f(x)]$ and $R[f(x)]$ both exist and also that both belong to $L_p(0, \infty)$.

When $r=1$, (3.2) and (3.3) reduces to the one given by Saxena (1967).

On the other hand if we set $m=0$, we obtain Erdélyi's (1940) operators, whereas $r=1$, $m=0$, give rise to Kober's (1940) operators.

The following results (Kalla and Saxena 1969) are useful in our investigations.

$$M\{I[\alpha, \beta, m, \gamma, \eta : f(x)]\} = \frac{\Gamma\left(\frac{1+\eta-s}{\gamma}\right)\left(\beta - \frac{1+\eta-s}{\gamma}\right)_m}{(\beta)_m\, \Gamma\left(1-\alpha+\frac{1+\eta-s}{\gamma}\right)} M\{f(x)\} \tag{3.4}$$

$$M\{R[\alpha, \beta, m, \gamma, \delta : f(x)]\} = \frac{\Gamma\left(\frac{\delta+s}{\gamma}\right)\left(\beta - \frac{s+\delta}{\gamma}\right)_m}{(\beta)_m \Gamma\left(1-\alpha+\frac{\delta+s}{\gamma}\right)} M\{f(x)\} \tag{3.5}$$

provided that the condition (3.1) holds.

Lemma 4—If (*i*) $x > 0$, (*ii*) $h(x) \in L_2(0, \infty)$, (*iii*) $\frac{e_j - h_j}{\lambda_j} \geqslant n_j$, (*iv*) $h_j + \frac{1}{2} > 0$, (*v*) $\frac{f_j + h_j}{\lambda_j} \neq 0, -1, -2, \ldots$; $n_j = 0, 1, 2, \ldots$; for $j = 1, 2, \ldots, r$ then

$$R_r\,[f(x)] \int_0^\infty R_{p,q,r}(xu)\, h(u)\, du = \int_0^\infty R_{p,q,r-1}(xu)\, h(u)\, du \tag{3.6}$$

where the contracted notation $R_r[f(x)]$ stands for

$$\left(\frac{f_r+h_r}{\lambda_r}\right)_{n_r} R\left[1+\frac{h_r-e_r}{\lambda_r}, \frac{f_r+h_r}{\lambda_r}, n_r, \lambda_r, h_r : f(x)\right]$$

and the integrals are considered as functions of x.

Proof: From (2.9) the integral on the left of (3.6) $\in L_2(0, \infty)$ and so we can apply the operator R to it by (3.1). It is also evident that the left-hand side of (3.6) belongs to $L_2(0, \infty)$ and therefore the operator M can be applied to it, by (2.3). Hence from (3.4) and (2.7) we obtain

$$M\{ R_r [f(x)] \int_0^\infty R_{p,q,r}(xu)\, h(u)\, du\} = \frac{\Gamma\left(\frac{h_r+s}{\lambda_r}\right)\Gamma\left(\frac{f_c-s}{\lambda_r}+n_r\right)}{\Gamma\left(\frac{e_r+s}{\lambda_v}\right)\Gamma\left(\frac{f_v-s}{\lambda_v}\right)} \chi(s)\, H(1-s)$$

$$= M\{ \int_0^\infty R_{p,q,r-1}(xu)\, h(u)\, du \}, \qquad (3.7)$$

From (2.9) and (3.1) the functions in (3.7) operated upon by M both belong to $L_2(0, \infty)$. Hence from (2.4) each side of (3.7) belong to $L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and so from (2.5) the operator M^{-1} can be applied to (3.7). This completes the proof of Lemma 4.

4. Inversion Formula

Theorem—If (*i*) $x > 0$, (*ii*) $h(x) \in L_2(0, \infty)$, (*iii*) $\left(\frac{e_j - h_j}{\lambda_j}\right) > n_j$

(*iv*) $h_j + \frac{1}{2} > 0$, (*v*) $\frac{f_j + h_j}{\lambda_j} \neq 0, -1, -2, \ldots$; $n_j = 0, 1, 2, \ldots$,

for $j = 1, 2, \ldots, r$; (*vi*) $\frac{d_i - b_i}{\mu_i} > m_i$, (*vii*) $\frac{b_i + c_i}{\mu_i} \neq 0, -1, -2, \ldots$;

$m_i = 0, 1, \ldots$, for $i = 0. 1, 2, \ldots, q$ and (*viii*) $a_k + \frac{1}{2} > 0$ for $k=0, 1, 2, \ldots, p$ and (*ix*) if $h(x)$ is solution of

$$\int_0^\infty R_{p,q,r}(xu)\, h(u)\, du = \phi(x)$$

which belongs to $L_2(0, \infty)$, then

$$h(x) = x^{\zeta_p - a_p - 1} \Big[\prod_{k=1}^{p} L^{-1} \Big\{ x^{-a_k} \prod_{i=1}^{q} \left(\frac{b_i + c_i}{\mu_i}\right)_{m_i}$$

$$I\left[1+\frac{h_j - d_j}{\mu_i}, \frac{b_i + c_i}{\mu_i}, m_i, \mu_i, b_i - 1 : \prod_{j=1}^{r} \left(\frac{f_j + h_j}{j}\right)_{n_j}\right.$$

$$\left. R\left(1+\frac{h_j - e_j}{\lambda_j}, \frac{f_j + h_j}{\lambda_j}, n_j, \lambda_j, h_j : \phi(x) \right)\right]\Big\}_{x=t^{1/\zeta_k}}\Big]_{u=x^{\zeta_k}} \qquad (4.1)$$

where L^{-1} denotes the inverse Laplace transform.

Proof: Lemma 4 may be regarded as a reduction formula for the integral in (1.7). The successive application of the operator R_j for $j=r$, $r-1,\ldots, 2, 1$, it is observed, from Lemma 4, that

$$\prod_{j=1}^{r}\left(\frac{f_j+h_j}{\lambda_j}\right)_{n_j} R\left[1+\frac{h_j-e_j}{\lambda_j},\ \frac{f_j+h_j}{\lambda_j},\ n,\lambda_j,h_j:\phi(x)\right]$$
$$=\int_0^\infty R_{p,q}(xu)\,h(u)\,du. \tag{4.2}$$

Similarly the successive application of the operator I_i for $i=q,\ldots 1$, transform the equation (4.2) in the form

$$\prod_{i=1}^{q}\left(\frac{b_i+c_i}{\mu_i}\right)_{m_i} I\left[1+\frac{b_i-d_i}{\mu_i},\ \frac{b_i+c_i}{\mu_i},\ m_i,\mu_i,b_i-1:\right.$$
$$\left.\prod_{j=1}^{r}\left(\frac{f_j+h_j}{\lambda_j}\right)_{n_j} R\left\{1+\frac{h_j-e_j}{\lambda_j},\ \frac{f_j+h_j}{\lambda_j},\ n_j,\mu_j,h_j:\phi(x)\right\}\right]$$
$$=\int_0^\infty R_p(xu)\,h(u)\,du \tag{4.3}$$

Now we must express the integral (4.3) in the form of a Mellin type integral and this can be done by using the Parseval Theorem (2.7). The application of (2.7) is justified by Titchmarsh (1937, p 60) with $\sigma-\frac{1}{2}$, if $f(x)$ and $f(x)\,x^{-1/2}$ both belong to $L(0,\infty)$ and $M\{f(x)\}$, the Mellin transform of $f(x)$ belong to $L(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$.

Multiplying both the sides by x^{-a_k} and changing x by t^{1/ζ_k} since the conditions of Lemma 2 are satisfied we can apply L^{-1} operator which cancels $\Gamma\left(\frac{a_k+s}{\zeta_k}\right)$ from the integrand. And the successive application of L^{-1} operator for $k=p, p-1, \ldots 1$, gives the right hand side

$$\frac{1}{2\pi i}\int_C F(1-s)\,u^{\frac{a_k+s}{\zeta_k}-1}\,ds. \tag{4.4}$$

Replacing u by x^{ζ_k} giving

$$\frac{x^{a_k-\zeta_k}}{2\pi i}\int_C F(1-s)\,x^s\,ds \tag{4.5}$$

on replacing s by $1-s$ in (4.5). If in (4.5), C is the line $\sigma=\frac{1}{2}$ this replacement leaves thc contour of integration unaltered.

Finally, an application of (2.2) to the right hand side of (4.5), yields (4.1).

5. Particular Cases

Case I—Fox's result (1963) can be obtained by taking $p=m_i=n_j=0$ for $i=1, 2, \ldots, q$, $j=1, 2, \ldots, r$ and $q=r$ with slight changing in the parameters of Gamma functions.

Case II—On the other hand if we take $m_i = n_j = 0$ for $i = 1, 2, \ldots, q$, $j = 1, 2, \ldots, r$, we get an inversion formula for a kernel consider by Saxena (1966) further if we take $r = 0$, we get an inversion formula for a kernel involving a Mellin-Bernes type integral considered by Saxena (1966).

Case III—Lastly if we put $p = r = \zeta_1 = \lambda_1 = 1$, $q = n_1 = 0$, $a_1 = a$, $e_1 = e$, $h_1 = h$, then it is found that

$$h(x) = x^{-a} L^{-1} \{ x^{-a} R[1+h-e, f_1+h, 0, 1, h : \phi(x)] \}$$

is the inversion formula for an integral transform, some times known as Verma transform,

$$(xu)^{\frac{1}{2}(a+e-1)} e^{-\frac{1}{2}(xu)} W_{l,k}, (xu)\, h(u)\, du = \phi(x)$$

where $l = \frac{1}{2}(1+a+e) - h$, $k = \frac{1}{2}(a-e)$,

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