# ELASTIC DEFORMATIONS DUE TO HEAT SOURCES IN FINITE CYLINDERS

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An exact analysis is presented for the deformation of a finite cylindrical elastic body when the temperature is distributed due to stationary heat source of unit intensity over the circumference of a circle situated inside the cylindrical body at its mid-plane. Moreover the author has assumed that the two end sections are thermally insulated while the cylindrical surface is subjected to mechanical loading.

## Nomenclature

$a$ = radius of the cylinder
$L$ = half length of cylinder
$\lambda$ = coefficient of internal heat conduction
$\mu$ = shear modulus
$\nu$ = Poisson's ratio
$\alpha_t$ = linear coefficient of thermal expansion
$u_r$, $u_z$ = displacement components
$e$ = volume dilatation
$\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$, $\sigma_{zr}$ = stress components
$T$ = temperature distribution
$J_\nu (kr)$, $(\nu=0, 1)$ = Bessel functions of the first kind with parameter $k$
$I_\nu (kr)$, $(\nu=0, 1)$ = modified Bessel functions of the first kind with parameter $k$
$\delta (r)$, $\delta (z)$ = Dirac-delta functions.

## 1. Introduction

The elastic as well as thermoelastic problems for finite dimensional bodies are becoming more important with the development of solid propellant rocket motors, shafts of machines, etc. Three dimensional thermoelastic problems usually deal with the case of elastic spaces, infinite cylinders and elastic layers (Nowacki 1962) though all the models are generally far away from realities (Nadeau 1964).

For the axially symmetric deformation of a solid cylinder of finite length exact solutions were obtained by Valov (1962) while Sun and Valanis (1966) have given an exact solution to the hollow finite length cylinder under mechanical loading. Sierakowski and sun (1968) have obtained an exact solution to the elastic deformation when a hollow cylindrical body of finite length is heated symmetrically about the midplane on the outer and inner surfaces. Here following their approach stresses and displacements have been obtained when the temperature is distributed due to a stationary heat source of unit intensity distributed over the circumference of a circle, situated inside an isotropic cylinder at its mid-plane, while the two end sections are thermally insulated, the cylindrical surface being kept at zero temperature. Mechanical normal loading has been applied on the cylindrical surface while the end sections are free of tractions.

## 2. Formulation of the Problem

We consider an isotropic finite circular cylinder of radius $a$ and length $2L$ with origin at the mid point of the axis of the cylinder which is taken as $z$-axix. Let us consider heat sources of unit intensity distributed over the circumference of a circle of the radius $\rho < a$ situated at the mid-plane of the cylinder. We have also assumed that the temperature vanishes on the surface ($r=a$) and the end sections are thermally insulated.

The temperature field due to heat source of unit intensity distributed over the circumference of a circle of radius $\rho$ at the mid-plane $z=0$ can be given by Dirac-delta function, that is

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial^2}{\partial z^2}\right] T = -\frac{1}{\lambda}\,\delta(r-\rho)\,\delta(z) \tag{2.1}$$

where $\lambda$ is the coefficient of heat conduction and $\delta(z)$, $\delta(r-\rho)$ are Dirac-delta functions. The temperature boundary conditions can be taken in the form

$$T(a, z) = 0$$
$$\left[\frac{\partial T}{\partial z}\right]_{z=\pm L}=0 \tag{2.2}$$

In the case of axisymmetric deformation of a solid under both thermal and mechanical loading, equilibrium equations reduce to

$$\nabla_1^2 u_r - \frac{u_r}{r^2}+\frac{1}{1-2\nu}\frac{\partial e}{\partial r}-\frac{2(1+\nu)}{(1-2\nu)}\alpha_t\frac{\partial T}{\partial z}=0$$
$$\nabla^2 u_z+\frac{1}{1-2\nu}\frac{\partial e}{\partial z}-\frac{2(1+\nu)}{(1-2\nu)}\alpha_t\frac{\partial T}{\partial z}=0 \tag{2.3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

$$e = e_{rr} + e_{\theta\theta} + e_{zz}.$$

On the surfaces of the cylinder the following boundary conditions have to be satisfied

$$\begin{aligned} \sigma_{rr}\Big]_{r=a} &= -p \\ \sigma_{zr}\Big]_{r=a} &= 0 \\ \sigma_{zr}\Big]_{z=\pm L} &= 0 \\ \sigma_{zz}\Big]_{z=\pm L} &= 0 \end{aligned} \tag{2.4}$$

## 3. Solution of the Problem

In the case of a finite body the Dirac-delta functions can be represented by the infinite series

$$\delta(r-\rho) = \frac{2\rho}{a^2}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\rho)\,J_0(\alpha_n r)}{J_1^2(\alpha_n a)}$$

$$\delta(z) = \frac{1}{2L} + \frac{1}{L}\sum_{m=1}^{\infty}\cos\beta_m z \tag{3.1}$$

where,

$$J_0(\alpha_n a) = 0,\ \beta_m = \frac{m\Pi}{L}.$$

Hence the solution of (2.1) with the help of the above equations can be taken as

$$T(r,z) = \frac{\rho}{\lambda L a^2}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\rho)\,J_0(\alpha_n r)}{\alpha_n^2 J_1^2(\alpha_n a)} + \frac{2\rho}{\lambda L a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\rho)\,J_0(\alpha_n r)}{(\alpha_n^2+\beta_m^2)\,J_1^2(\alpha_n a)}. \tag{3.2}$$

Introducing the displacement potential function $\bar{\phi}$ such that

$$u_r = \frac{\partial\bar{\phi}}{\partial r},\qquad u_z = \frac{\partial\bar{\phi}}{\partial z},$$

eqns (2.3) reduce to

$$\nabla^2\bar{\phi} = \frac{\rho m_0}{\lambda L a^2}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\rho)\,J_0(\alpha_n r)}{\alpha_n^4 J_1^2(\alpha_n a)} + \frac{2\rho m_0}{\lambda L a^2}\sum_{m=1}^{\infty}\sum_{n=8}^{\infty}\frac{J_0(\alpha_n\rho)\,J_0(\alpha_n r)}{(\alpha_n^2+\beta_m^2)^2 J_1^2(\alpha_n a)}\cos\beta_m z, \tag{3.3}$$

and a particular solution with $\bar{\phi} = 0$, $\frac{\partial \bar{\phi}}{\partial z} = 0$ (on $r = a, z = \pm L$, respectively), can be taken as

$$\bar{\phi} = -\frac{\rho m_0}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) J_0(\alpha_n r)}{\alpha_n^2 J_1^2(\alpha_n a)} - 2\frac{\rho m_0}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) J_0(\alpha_n r)}{(\alpha_n^2 + \beta_m^2) J_1^2(\alpha_n a)} \cos \beta_m z \tag{3.4}$$

where

$$m_0 = \frac{1+\nu}{1-\nu} \alpha_t.$$

Now the stress components due to thermoelastic distribution can be given by (Nowacki 1962)

$$\bar{\sigma}_{rr} = -2\mu(\bar{\phi}_{,zz} + r^{-1}\bar{\phi}_{,r})$$

$$\bar{\sigma}_{\theta\theta} = -2\mu(\bar{\phi}_{,zz} + \bar{\phi}_{,rr})$$

$$\bar{\sigma}_{zz} = -2\mu(\bar{\phi}_{,rr} + r^{-1}\bar{\phi}_{,r})$$

$$\bar{\sigma}_{zr} = 2\mu\bar{\phi}_{,zr}$$

i. e.

$$\bar{\sigma}_{rr} = -\frac{2\mu m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) J_1(\alpha_n r)}{r \alpha_n^3 J_1^2(\alpha_n a)} - \frac{4\mu m_0 \rho}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\{\beta_m^2 J_0(\alpha_n r) - (\alpha_n / r) J_1(\alpha_n r)\}}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z$$

$$\bar{\sigma}_{\theta\theta} = -\frac{2\mu m_0 \rho)}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\{J_0(\alpha_n r) - (1/\alpha_n r) J_1(\alpha_n r)\}}{\alpha_n^2 J_1^2(\alpha_n a)} - \frac{4\mu m_0 \rho}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\{(\alpha_n^2 + \beta_m^2) J_0(\alpha_n r) - (\alpha_n / r) J_1(\alpha_n r)\}}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z$$

$$\bar{\sigma}_{zz} = -\frac{2\mu m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) J_0(\alpha_n r)}{\alpha_n^2 J_1^2(\alpha_n a)} - \frac{4\mu m_0 \rho}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n \rho) J_0(\alpha_n r)}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z$$

$$\bar{\sigma}_{zr} = -\frac{4\mu m_0 \rho}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_n \beta_m J_0(\alpha_n \rho) J_1(\alpha_n r)}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \sin \beta_m z.$$

We observe that on the plane bases $z = \pm L$, the shearing stress component $\bar{\sigma}_{zr}$ vanishes but the normal stress component $\bar{\sigma}_{zz}$ is non-zero similarly the shearing stress component is non-zero on the lateral surface of the cylinder. With a view to satisfy all the stress boundary conditions on the end sections as well as on the lateral surface, we impose complementary stress systems over the systems already considered. The complimentary stress systems just cited above, can be expressed in terms of Love functions which are well known for the case of orthotropic bodies by satisfying the biharmonic equations

$$\nabla^2 \nabla^2 \bar{\bar{\phi}} = 0, \quad \nabla^2 \nabla^2 \bar{\bar{\bar{\phi}}} = 0. \tag{3.5}$$

where the stress system can be given by

$$\left(\bar{\bar{\sigma}}_{rr}, \bar{\bar{\bar{\sigma}}}_{rr}\right) = \frac{2\mu}{1-2\nu} \partial_z \left[\nu \nabla^2 - \partial_r^2\right] \left(\bar{\bar{\phi}}, \bar{\bar{\bar{\phi}}}\right)$$

$$\left(\bar{\bar{\sigma}}_{\theta\theta}, \bar{\bar{\bar{\sigma}}}_{\theta\theta}\right) = \frac{2\mu}{1-2\nu} \partial_z \left[\nu \nabla^2 - r^{-1} \partial_r\right] \left(\bar{\bar{\phi}}, \bar{\bar{\bar{\phi}}}\right)$$

$$\left(\bar{\bar{\sigma}}_{zz}, \bar{\bar{\bar{\sigma}}}_{zr}\right) = \frac{2\mu}{1-2\nu} \partial_z \left[(2-\nu) \nabla^2 - \partial_z^2\right] \left(\bar{\bar{\phi}}, \bar{\bar{\bar{\phi}}}\right)$$

$$\left(\bar{\bar{\sigma}}_{zr}, \bar{\bar{\bar{\sigma}}}_{zr}\right) = \frac{2\mu}{1-2\nu} \partial_r \left[(1-\nu) \nabla^2 - \partial_z^2\right] \left(\bar{\bar{\phi}}, \bar{\bar{\bar{\phi}}}\right). \tag{3.6}$$

We take the solutions of (3.5) in the form

$$\bar{\bar{\phi}} = \sum_{m=1}^{\infty} \left[A_m I_0 (\beta_m^r) + B_m \beta_m^r I_1 (\beta_m^r)\right] \frac{1}{\beta_m^3} \sin \beta_m^z$$

$$\bar{\bar{\bar{\phi}}} = \sum_{k=1}^{\infty} \left[C_k \sinh \lambda_k z + D_k \lambda_k z \cosh \lambda_k z\right] \frac{1}{\lambda_k^2} J_0 (\lambda_k r)$$

where $A_m$, $B_m$, $C_k$, $D_k$ are independent of variables $r$ and $z$ and $\lambda_k$ are the eigen constants. In addition to these systems the following complementary solutions satisfying eqns. (2.3) have also been considered for completeness

$$u_r^0 = G \,.\, r, \quad u_z^0 = H \,.\, z \tag{3.7}$$

where $G$ and $H$ are constants to be determined.

Hence the total stresses and displacements in terms of the arbitrary constants $A_m$, $B_m$, $C_k$, $D_k$ can be written as

$$\sigma_{rr} = \sigma_{rr}^0 + \bar{\sigma}_{rr} + \bar{\bar{\sigma}}_{rr} + \bar{\bar{\bar{\sigma}}}_{rr}$$

$$= \frac{2\mu}{1-2\mu} G + \frac{2\mu\nu}{1-2\nu} H - \frac{2\mu\, m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\, J_1(\alpha_n r)}{\alpha_n^3 r\, J_1^2(\alpha_n a)}$$

$$- \frac{4\mu\, m_0\, \rho}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\left[\beta_m^2 J_0(\alpha_n r) - (\alpha_n / r) J_1(\alpha_n r)\right]}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{m=1}^{\infty} \Big[ (2\nu - 1) B_m I_0(\beta_m r) - B_m \beta_m r\, I_1(\beta_m r)$$

$$- A_m I_0(\beta_m r) + \frac{A_m}{\beta_m r} I_1(\beta_m r)\Big] \cos \beta_m z +$$

$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty} \Big[ 2\nu D_k \lambda_k \cosh \lambda_k z\, J_0(\lambda_k r) + \{\lambda_k J_0(\lambda_k r) - \frac{1}{r} J_1(\lambda_k r)\}$$

$$\{c_k \cosh \lambda_k z + D_k \lambda_k z \sinh \lambda_k z + D_k \cosh \lambda_k z\}\Big] \tag{3.8}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{0} + \bar{\sigma}_{\theta\theta} + \bar{\bar{\sigma}}_{\theta\theta} + \bar{\bar{\bar{\sigma}}}_{\theta\theta}$$

$$= \frac{2\mu}{1-2\nu} G + \frac{2\mu\nu}{1-2\nu} H - \frac{2\mu m_0 \rho}{\lambda La^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\{J_0(\alpha_n r) - (1/\alpha_n r) J_1(\alpha_n r)\}}{\alpha_n^2 J_1^2(\alpha_n a)} -$$

$$- \frac{4\mu m_0 \rho}{\lambda La^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\{(\alpha_n^2 + \beta_m^2) J_0(\alpha_n r) - (\alpha_n/r) J_1(\alpha_n r)\}}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z +$$

$$+ \frac{2\mu}{1-2\nu} \sum_{m=1}^{\infty} \left[ (2\nu - 1) B_m I_0(\beta_m r) - \frac{A_m}{\beta_m r} I_1(\beta_m r) \right] \cos \beta_m z +$$

$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty} \left[ \frac{1}{r} J_1(\lambda_k r) \{ c_k \cosh \lambda_k z + D_k \lambda_k z \sinh \lambda_k z + D_k \cosh \lambda_k z \right.$$

$$\left. + 2\nu D_k \lambda_k \cosh \lambda_k z \, J_0(\lambda_k r) \right] \tag{3.9}$$

$$\sigma_{zz} = \sigma_{zz}^{0} + \bar{\sigma}_{zz} + \bar{\bar{\sigma}}_{zz} + \bar{\bar{\bar{\sigma}}}_{zz}$$

$$= \frac{4\mu\nu}{1-2\nu} G + \frac{2\mu(1-\nu)}{1-2\nu} H - \frac{2\mu m_0 \rho}{\lambda La^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) J_0(\alpha_n r)}{\alpha_n^2 J_1^2(\alpha_n a)}$$

$$- \frac{4\mu m_0 \rho}{\lambda La^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n \rho) J_0(\alpha_n r)}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{m=1}^{\infty} \left[ 2(2-\nu) B_m I_0(\beta_m r) + B_m \beta_m r I_1(\beta_m r) + \right.$$

$$\left. + A_m I_0(\beta_m r) \right] \cos \beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty} \left[ (1-2\nu) D_k \lambda_k \cosh \lambda_k z - c_k \lambda_k \cosh \lambda_k z \right.$$

$$\left. - D_k \lambda_k^2 z \sinh \lambda_k z \right] J_0(\lambda_k r) \tag{3.10}$$

$$\sigma_{zr} = \sigma_{zr}^{0} + \bar{\sigma}_{zr} + \bar{\bar{\sigma}}_{zr} + \bar{\bar{\bar{\sigma}}}_{zr}$$

$$= -\frac{4\mu m_0 \rho}{\lambda La^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_n \beta_m J_0(\alpha_n \rho) J_1(\alpha_n r)}{(\alpha_n^2 + \beta_m^2)^2 J_1^2(\alpha_n a)} \sin \beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{m=1}^{\infty} \left[ 2(1-\nu) B_m I_1(\beta_m r) + B_m \beta_m^2 r I_0(\beta_m r) \right.$$

$$\left. + A_m \beta_m I_1(\beta_m r) \right] \sin \beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty} \left[ 2\nu \lambda_k D_k \sinh \lambda_k z + c_k \lambda_k \sinh \lambda_k z \right.$$

$$\left. + D_k \lambda_k^2 z \cosh \lambda_k z \right] J_1(\lambda_k r) \tag{3.11}$$

while the displacements are

$$u_r = u_r^0 + \bar{u}_r + \bar{\bar{u}}_r + \bar{\bar{\bar{u}}}_r$$

$$= G.r + \frac{m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho) J_1(\alpha_n r)}{\alpha_n^3 J_1^2(\alpha_n a)}$$

$$+ \frac{2\rho m_0}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_n J_0(\alpha_n \rho) J_1(\alpha_n r)}{(\alpha_n^2+\beta_m^2)^2 J_1^2(\alpha_n a)} \cos \beta_m z$$

$$- \frac{1}{1-2\nu} \sum_{m=1}^{\infty} \left[ \frac{A_m}{\beta_m r} I_1(\beta_m r) + B_m I_0(\beta_m r) \right] r \cos \beta_m z$$

$$+ \frac{1}{1-2\nu} \sum_{k=1}^{\infty} \left[ c_k \cosh \lambda_k z + D_k \cosh \lambda_k z + D_k \lambda_k z \sinh \lambda_k z \right] J_1(\lambda_k r) \tag{3.12}$$

$$uz = u_z^0 + \bar{u} + \bar{\bar{u}} + \bar{\bar{\bar{u}}}_z$$

$$= H.z + \frac{2\rho m_0}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{u=1}^{\infty} \frac{\beta_m J_0(\alpha_n \rho) J_0(\alpha_n r)}{(\alpha_n^2+\beta_m^2)^2 J_1^2(\alpha_n a)} \sin \beta_m z$$

$$+ \frac{1}{1-2\nu} \sum_{m=1}^{\infty} \frac{1}{\beta_m} \Big[ 4(1-\nu) B_m I_0(\beta_m r) + A_m I_0(\beta_m r)$$

$$+ B_m \beta_m r I_1(\beta_m r) \Big] \sin \beta_m z$$

$$+ \frac{1}{1-2\nu} \sum_{k=1}^{\infty} \Big[ 2(1-2\nu) D_k \sinh \lambda_k z - C_k \sinh \lambda_k z$$

$$- D_k \lambda_k z \cosh \lambda_k z \Big] J_0(\lambda_k r). \tag{3.13}$$

## 4. Determination of Arbitrary Constants

Now using the third boundary conditions of (2.4) we get

$$C_k = -[2\nu + \lambda_k L \coth \lambda_k L] D_k, \tag{4.1}$$

and from the second condition of (2.4)

$$J_1(\lambda_k a) = 0, \tag{4.2}$$

which determines $\lambda_k$ in radial direction. Here we have neglected the solution $C_k=0$, $D_k=0$, which is true only in the case of infinite cylinders, and also we get

$$[2(1-\nu) I_1(\beta_m a) + \beta_m^2 a I_0(\beta_m a)] B_m + \beta_m I_1(\beta_m a) Am$$

$$= \frac{2(1-2\nu) m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{\alpha_n \beta_m J_0(\alpha_n \rho)}{(\alpha_n^2+\beta_m^2)^2 J_1(\alpha_n a)}. \tag{4.3}$$

Using the remaining boundary conditions of (2.4) we get the following relations :

$$\frac{2\mu}{1-2\nu} G + \frac{2\mu\nu}{1-2\nu} H - \frac{2\mu m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)}{\alpha_n^3 a J_1(\alpha_n a)}$$

$$+ \frac{4\mu\rho m_0}{\lambda L a^3} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\alpha_n J_0(\alpha_n \rho)}{(\alpha_n^2+\beta_m^2)^2 J_1(\alpha_n a)} \cos\beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{m=1}^{\infty}\Big[(2\nu-1) B_m I_0(\beta_m a) - B_m \beta_m a I_1(\beta_m a)$$

$$- A_m I_0(\beta_m a) + \frac{A_m}{\beta_m a} I_1(\beta_m a)\Big] \cos\beta_m z$$

$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty}\Big[(1-\lambda_k L \coth\lambda_k L)\cosh\lambda_k z$$

$$+ \lambda_k z \sinh\lambda_k z\Big] D_k \lambda_k J_0(\lambda_k a)$$

$$= -p. \tag{4.4}$$

and

$$\frac{4\mu\nu}{1-2\nu} G + \frac{2\mu(1-\nu)}{1-2\nu} H - \frac{2\mu m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n\rho) J_0(\alpha_n r)}{\alpha_n^2 J_1^2(\alpha_n a)}$$

$$- \frac{4\mu m_0 \rho}{\lambda L a^2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-1)^m \frac{\alpha_n^2 J_0(\alpha_n\rho) J_0(\alpha_n r)}{(\alpha_n^2+\beta_n^2)^2 J_1^2(\alpha_n a)} +$$

$$+ \frac{2\mu}{1-2\nu} \sum_{m=1}^{\infty} (-1)^m \Big[2(2-\nu) B_m I_0(\beta_m r) + B_m \beta_m r I_1(\beta_m r) + A_m I_0(\beta_m r)\Big]$$

$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty} D_k \lambda_k \Big[\cosh\lambda_k L + \lambda_k L \,\mathrm{cosech}\,\lambda_k L\Big] J_0(\lambda_k r)$$

$$= 0. \tag{4.5}$$

We introduce the fiinite Fourier cosine transforms and represent all the Bessel functions in terms of zeroth order Bessel function by

$$I_0(\beta_m r) = \overset{*}{I}_{0,m} + \sum_{k=1}^{\infty} \overset{*}{I}_{0,mk} J_0(\lambda_k r)$$

$$r I_1(\beta_m r) = \overset{*}{I}_{1,m} + \sum_{k=1}^{\infty} \overset{*}{I}_{1,mk} J_0(\lambda_k r)$$

$$J_0(\alpha_n r) = \overset{*}{J}_{0,n} + \sum_{k=1}^{\infty} \overset{*}{J}_{0,nk} J_0(\lambda_k r)$$

$$d_1 \cosh\lambda_k z + d_2 z \sinh\lambda_k z = \frac{\overset{*}{a}_{1,k}}{2L} + \frac{1}{L}\sum_{m=1}^{\infty} \overset{*}{a}_{1,km} \cos\beta_m z \tag{4.6}$$

where

$$d_1 = (1-\lambda_k L \coth \lambda_k L)\, \lambda_k\, J_0\,(\lambda_k a)$$

$$d_2 = \lambda_k^2\, I_0\,(\lambda_k a).$$

The coefficients of the above expansions can be evaluated by using the orthogonality conditions viz.

$$\overset{*}{a}_{1,k} = \int_{-L}^{L} (d_1 \cosh\ \lambda_k z + d_2 z \sinh \lambda_k z)\, dz$$

$$\overset{*}{a}_{1,km} = \int_{-L}^{L} (d_1 \cosh \lambda\ z + d_2 z \sinh \lambda_k z) \cos \beta_m z\, dz$$

$$\overset{*}{I}_{0,m} = \frac{2}{a^2} \int_0^a r\, I_0\,(\beta_m r)\, dr$$

$$\overset{*}{I}_{1,m} = \frac{2}{a^2} \int_0^a r^2\, I_1\,(\beta_m\, r)\, dr$$

$$\overset{*}{J}_{0,n} = \frac{2}{a^2} \int_0^a r\, \mathrm{J}_0\,(\alpha_n r)\, dr$$

$$\overset{*}{I}_{0,mk} = \frac{2}{a^2 \mathrm{J}_0\,(\lambda_k a)} \int_0^a r\, I_0\,(\beta_m r)\, J_0\,(\lambda_k r)\, dr$$

$$\overset{*}{I}_{1,mk} = \frac{2}{a^2 \mathrm{J}_0\,(\lambda_k a)} \int_0^a r^2 I_1\,(\beta_m\, r)\, J_0\,(\lambda_k r)\, dr$$

$$\overset{*}{J}_{1,nk} = \frac{2}{a^2 J_0\,(\lambda_k a)} \int_0^a r\, J_0\,(\alpha_n r)\, J_0\,(\lambda_k r)\, dr$$

Substituting the above transformations in (4.4) and (4.5) and equating like terms we get the following expressions

$$\frac{2\mu}{1-2\nu}\, G + \frac{2\mu\nu}{1-2\nu}\, H - \frac{2\mu m_0 \rho}{\lambda L a^2} \sum_{n=1}^{\infty} \frac{J_0\,(\alpha_n \rho)}{\alpha_n^2\, a\, J_1\,(\alpha_n a)} +$$
$$+ \frac{2\mu}{1-2\nu} \sum_{k=1}^{\infty} \frac{a^*_{1,k}}{2L}\, D_k + p = 0$$

$$\frac{2\nu}{1-2\nu}\, G + \frac{1-\nu}{1-2\nu}\, H - \frac{m_0 \rho}{\lambda L a^2} \sum_{n=1}^{8} \frac{J_0\,(\alpha_n \rho)}{\alpha_n^2\, J_1^2\,(\alpha_n a)}\, \overset{*}{J}_{0,n} +$$
$$- \frac{2 m_0 \rho}{\lambda L a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^m\, \alpha_n^2\, J_0\,(\alpha_n \rho)}{(\alpha_n^2 + \beta_m^2)^2\, J_1^2\,(\alpha_n a)}\, \overset{*}{J}_{0,n} +$$
$$+ \frac{1}{1-2\nu} \sum_{m=1}^{\infty} (-1)^m \left[ 2\,(2-\nu)\, B_m\, \overset{*}{I}_{0,m} + B_m\, \beta_m\, \overset{*}{I}_{1,m} + A_m\, \overset{*}{I}_{0,m} \right]$$
$$= 0. \tag{4.7}$$

$$-\frac{2m_0\rho}{\lambda La^3}\sum_{n=1}^{\infty}\frac{\alpha_n J_0(\alpha_n\rho)}{(\alpha_n^2+\beta_m^2)^2 J_1(\alpha_n a)}+\frac{1}{L(1-2\nu)}\sum_{k=1}^{\infty}\overset{*}{a}_{1,km} D_k$$

$$+\frac{1}{1-2\nu}\Big[(2\nu-1) B_m I_0(\beta_m a)-B_m \beta_m aI_1(\beta_m a)$$

$$-A_m I_0(\beta_m a)+\frac{A_m}{\beta_m a} I_1(\beta_m a)\Big]$$

$$=0, (m=1, 2, 3,\dots \dots) \tag{4.7a}$$

$$-\frac{m_0\rho}{\lambda La^2}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n\rho)}{\alpha_n^2 J_1^2(\alpha_n a)}\overset{*}{J}_{0,nk}$$

$$-\frac{2m_0\rho}{\lambda La^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^m\alpha_n^2 J_0(\alpha_n\rho)}{(\alpha_n^2+\beta_m^2)^2 J_1^2(\alpha_n a)}\overset{*}{J}_{0,nk}$$

$$+\frac{1}{1-2\nu}\sum_{m=1}^{\infty}(-1)^m\Big[2(2-\nu) B_m \overset{*}{I}_{0,mk}+B_m \beta_m \overset{*}{I}_{1,mk}+A_m \overset{*}{I}_{0,mk}\Big]$$

$$+\frac{1}{1-2\nu}\Big[\cosh \lambda_k L+\lambda_k L \ \mathrm{cosech}\ \lambda_k L\Big]\lambda_k D_k$$

$$=0, (k=1, 2, 3,\dots\dots). \tag{4.8}$$

From (4.3) and (4.7*a*) we have

$$B_m=\frac{1}{S_1 S_5-S_2 S_4}\Big[S_2S_6-S_3S_5+\frac{S_2}{L}\sum_{k=1}^{\infty}\overset{*}{a}_{1,km}D_k\Big] \tag{4.9}$$

$$A_m=\frac{1}{S_1 S_5-S_2 S_4}\Big[S_3 S_4-S_1 S_6-\frac{S_1}{L}\sum_{k=1}^{\infty}\overset{*}{a}_{1,km} D_k\Big] \tag{4.10}$$

where

$$S_1=2(1-\nu) I_1(\beta_m a)+\beta_m^2 a I_0(\beta_m a)$$

$$S_2=\beta_m I_1(\beta_m a)$$

$$S_3=-\frac{2(1-2\nu) m_0\rho}{\lambda La^2}\sum_{n=1}^{\infty}\frac{\alpha_n \beta_m J_0(\alpha_n\rho)}{(\alpha_n^2+\beta_m^2)^2 J_1(\alpha_n a)}$$

$$S_4=(2\nu-1) I_0(\beta_m a)-\beta_m a I_1(\beta_m a)$$

$$S_5=\frac{1}{a\beta_m} I_1(\beta_m a)-I_0(\beta_m a)$$

$$S_6=\frac{2(1-2\nu) m_0\rho}{\lambda La^3}\sum_{n=1}^{\infty}\frac{\alpha_n J_0(\alpha_n\rho)}{(\alpha_n^2+\beta_m^2)^2 J_1(\alpha_n a)}.$$

Substituting (4.9) and (4.10) in (4.8) we get a set of simultaneous equations for determining $D_k$ as follows :

$$S_7 D_k = \frac{(1-2\nu)\,\rho m_0}{\lambda La^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)}{\alpha_n^2 J_1^2(\alpha_n a)} J^*_{0,nk} +$$

$$+ \frac{2(1-2\nu) m_0 \rho}{\lambda La^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^m \alpha_n^2 J_0(\alpha_n \rho)}{(\alpha_n^2+\beta_m^2)^2 J_1^2(\alpha_n a)} J^*_{0,nk} -$$

$$- \sum_{m=1}^{\infty} \frac{(-1)^m}{S_1 S_5 - S_2 S_4} \left[ 2(2-\nu) I^*_{0,mk} + \beta_m I^*_{1,mk} \right]$$

$$\left[ S_2 S_6 - S_3 S_5 + \frac{S_2}{L} \sum_{i=1}^{\infty} a^*_{1,im} D_i \right] -$$

$$- \sum_{m=1}^{\infty} \frac{(-1)^m}{S_1 S_5 - S_2 S_4} I^*_{0,mk}$$

$$\left[ S_3 S_4 - S_1 S_6 - \frac{S_1}{L} \sum_{i=1}^{8} a^*_{i,lm} D_i \right]$$

$$(k=1, 2, 3, \ldots) \qquad (4.11)$$

where

$$S_7 = \lambda_k \left[\cosh \lambda_k L + \lambda_k L \operatorname{cosech} \lambda_k L\right].$$

The remaining unknown constants $G$ and $H$ are then easily obtained from eqn (4.1) and (4.2). It is interesting to note that in satisfying the end conditions the constants $G$ and $H$ now depend on the constants $D_k$, $A_m$, $B_m$.

## 5. Concluding Remarks

The above type of infiinte systems of equation (4.11) are shown to be convergent by Valov (1962). For a particutar problem a finite number of terms of the above series are to be retained and computers have to be employed.

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