ON VECTORIAL PARANORMS

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Concept of a vectorial paranorm as a mapping from a finite dimensional vector space X into the positive cone R^k_+ has been introduced. The relationship between vectorial pseudonorm and vectorial paranorm is established. Regularity, totality and dual of a vectorial paranorm are discussed besides establishing various properties of vectorial paranorms.

1. INTRODUCTION

Let R^k_+ denote the set of all k-tuples of non-negative real numbers partially ordered componentwise; $\mathbb{C}^n$ denotes the vector space of all n-tuples of complex numbers.

Kantorovich (1939) introduced the concept of a vectorial norm as a mapping from the vector space $\mathbb{C}^n$ into the positive cone of R^k_+. A vectorial norm of order k on $\mathbb{C}^n$ is a mapping p: $\mathbb{C}^n \to R^k_+$ such that

$$p(ax) = |a| \; p(x), \forall \; x \in \mathbb{C}^n, \forall \; a \in \mathbb{C}. \qquad (1\cdot1)$$

$$p(x+y) \leqslant p(x)+p(y), \quad \forall \quad x,y \in \mathbb{C}^n. \qquad (1\cdot2)$$

$$p(x) \neq 0 \text{ if } x \neq 0. \qquad (1\cdot3)$$

Vectorial norms have also been studied by Robert (1964, 1965), Stoer (1968), Fielder and Pták (1962). Deutsch (1971) introduced vectorial pseudonorm of order k on $\mathbb{C}^n$ which is a mapping p: $\mathbb{C}^n \to R^k_+$ satisfying the axioms (1·1) and (1·2).

In this paper we introduce a much more general concept of vectorial paranorm of order k.

Definition (1·1)—Let X be a finite dimensional vector space over the real or complex field. A map f: $X \to R^k_+$ is a vectorial paranorm of order k, if the following conditions are satisfied.

(i) $f(0) = 0$

(ii) $f(-x) = f(x), \qquad \forall \; x \in X$

(iii) $f(x+y) \leqslant f(x) + f(y), \; \forall \; x,y \in X$

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(iv) If $\{t_n\}$ is a sequence of scalars such that $t_n \rightarrow t$, and $\{x_n\}$ is a sequence of vectors such that $x_n \rightarrow x$, then $f(x_n t_n - xt) \rightarrow 0$ (continuity of multiplication).

It is observed that for $k=1$ our vectorial paranorm coincides with the usual concept of paranorm or quasinorm. (See Wilansky 1964, Yosida 1965).

A vector space X on which there is defined a vectorial paranorm of order k will be called vectorial paranormed space.

Definition 1.2—A vectorial paranorm 'f' of order k, for which $f(x)=0 \Rightarrow x=0$ will be called total.

We shall abbreviate vectorial paranorm as *VP* and total vectorial paranorm as *TVP*.

2. Properties of Vectorial Paranorms

For the following we consider, f, f' as the *VP* of order k, $f_1, f_2, \ldots, f_k$ the components of f, x, y, x_n as the vectors and α, α_n as the scalars.

Proposition 2.1—(1) : $f(x \pm y) \geqslant |f(x) - f(y)|$

(2) : $f(x) \geqslant 0, \forall x \in X$.

(3) : If $\{\alpha_n\}$ is bounded and $f(x_n) \rightarrow 0$, then $f(\alpha_n x_n) \rightarrow 0$.

(4) : For any positive integer n, $f(nx) \leqslant n f(x)$, $f\left(\frac{x}{n}\right) \geqslant \left(\frac{f(x)}{n}\right)$

(5) $f^{\perp} = \{x \in X : f(x) = 0\}$ is a linear subspace of X.

(6) For any x, f is constant on the affine set $x + f^{\perp}$.

(7) If there exists a number M such that $f'(x) \leqslant M f(x)$, then f is stronger than f'. The converse is not true.

(8) Suppose that f is zero on a linear subspace S, and that $f(x+S)$ is defined to be $f(x)$, then f is a *VP* on X/S and it is *TVP* on $X/S^{\perp}$.

(9) If f, f' are both zero on a linear subspace S and f is stronger than f' on X/S. Then f is stronger than f'.

(10) The set $\{x \in X : f(x) <\}\epsilon$ is absorbing for $\epsilon > 0$.

Proof : (1) We have, $f(x) = f(x+y-y) \leqslant f(x+y) + f(-y)$

$$= f(x+y) + f(y).$$

Hence, $f(x) - f(y) \leqslant f(x+y)$.

Similarly, we get, $f(y) - f(x) \leqslant f(x+y)$,

which gives, $|f(x)-f(y)| \leqslant f(x+y)$.
Similarly result for $f(x-y)$ also follows.

(2) From (1) we have, for $y=0$, $|f(x)| \leqslant f(x)$.
But $f(x) \leqslant |f(x)|$ is always true.
We get $|f(x)| = f(x)$ which implies $f(x) \geqslant 0$.

(3) Suppose the contrary, i.e.
$\exists \ \epsilon > 0$ such that
$f(a_n x_n) > \epsilon$ for infinitely many n.
Now, from the sequence $\{n\}$ let us consider a subsequence $\{n(k)\}$ such that $a_{n(k)} \to a$. Then $f(x_{n(k)}) \to 0$ and $f(t_{n(k)} x_{n(k)}) \to 0$ (because of continuity of multiplication) leading to a contradiction.

(4) $f(nx) = f(x+x+..+x)$, (n times)
$\leqslant f(x)+..+f(x)$,
$= nf(x)$.

Hence, $nf\left(\frac{x}{n}\right) \geqslant f\left(n.\frac{x}{n}\right) = f(x)$.

Therefore, $f\left(\frac{x}{n}\right) \geqslant \frac{f(x)}{n}$.

(5) For any scalar a, $f(ax)=0$. Hence $ax \in f^{\perp}$.
Also, if $x, y \in f^{\perp}$, $f(x+y) \leqslant f(x) + f(y)$
implies, $x+y \in f^{\perp}$. Hence, $f^{\perp}$ is a linear subspace of X.

(6) Let $y \in x + f^{\perp}$. By (1) we have,
$f(y-x) \geqslant |f(y) - f(x)|$ and $y-x \in f^{\perp}$, i.e. $f(y-x) = 0$.
Hence, $|f(y) - f(x)| \leqslant 0$ implies $f(x) = f(y)$.

(7) If $f(x) \longrightarrow 0$ then $\text{M} f(x) \longrightarrow 0$ and by hypothesis $f'(x) \longrightarrow 0$. Hence f is stronger than f'.

Remark—Let $f(x,y) = \dfrac{|x|}{(1+|x|)+|y|}$ and $f'(x,y) = |x|$.

Then f is stronger than f'. But for $y=0$,
$f'(x,0) = |x| > \text{M}$, (where M is a constant).

Then $M > M \dfrac{|x|}{1+|x|} = M \; f(x,0)$,

i.e. $f'(x,0) > M f(x,0)$.

This implies that the converse is not true.

(8) By definition of f, we have, $f(0+S) = f(0) = 0$,
$f(x+S) = f(x) \geqslant 0, \forall \; x \in X$.
Also $f(-x+S), = f(x+S), \; f(x+y+S) \leqslant f(x+S) + f(y+S)$ and
$f[a_n(x_n+S) - a(x+S)] = f(a_n x_n - a\,x) \longrightarrow 0$.
Hence f is a *VP*. If, $f(x+S) = 0$, then $f(x)=0$. Hence $x \in S$,
so that $x + S = S$. Thus, f is a *TVP* on $X/S^{\perp}$.

(9) Let $f(x_n) \longrightarrow 0$. Then $f(x_n + S) \longrightarrow 0$ and
$f'(x_n + S) \longrightarrow 0$, i.e. $f'(x_n) \longrightarrow 0$.

(10) Let x be any vector. If it is not absorbed by the given
set, $\exists$ a sequence $\{a_n\}$ of scalars with $|a_n| \longrightarrow \infty$ and
$x \notin a_n . \{x \in X : f(x) < \epsilon\}$ i.e.
$x/a_n \notin \{x \in X : f(x) < \epsilon\}$. This implies
$f(x/a_n) > \epsilon$, contradicting the definition of f. Hence, the given set is absorbing.

Proposition 2.2—f is a *VP* iff each f_i $(i=1,..,k)$ is a paranorm in the functional analytic sense.

PROOF : The if part is obvious.

Conversely, let f be a *VP*. It is trivial to show that each f_i $(i=1,..,k)$ satisfies axioms (i) to (iii) of definition (1·1). We need only show continuity of multiplication for each f_i. If $\{a_n\}$ is a sequence of scalars with $a_n \to a$ and $\{x_n\}$ is a sequence of vectors with $x_n \to x$ then $f(a_n x_n - ax) \to 0$ implies, $[f_1(a_n x_n - ax), .., f_k(a_n x_n - ax)] \to 0$ i.e. $f_i(a_n x_n - ax) \to 0$, $(i=1,.., k)$. The proof is complete.

Theorem 2.1—Let $\{f^k\}$ be a sequence of *VP* on a linear subspace.

$$\text{Let } f(x) = \sum_{k=1}^{\infty} b_k \cdot \frac{f^k(x)}{1+f^k(x)} \tag{2·1}$$

such that

$\sum\limits_{k=1}^{\infty} b_k < \infty$ (i.e. Fre'chet combination of *VP's*).

Then, (a) f is a *VP* and satisfies

$f(x_n) \to 0$ iff $f^k(x_n) \to 0$ for each k. .. .. .. (2·2)

(b) f is the weakest *VP* which is stronger than each f^k.

(c) f is total iff $\{f^k\}$ is a total set.

PROOF : (a) For the first part we need only prove (iv) of definition (1·1) Let $\{\alpha_n\}$ be a sequence of scalars with $\alpha_n \to \alpha$ and $\{x_n\}$ be a sequence of vectors with $x_n \to x$. Then,

$$f(\alpha_n x_n - \alpha x) = \sum_{k=1}^{\infty} b_k \cdot \frac{f^k(\alpha_n x_n - \alpha x)}{1+f^k(\alpha_n x_n - \alpha x)} \to \sum_{k=1}^{\infty} b_k \cdot \frac{0}{1+0} = 0.$$

Hence f is a *VP*.

To prove (2·2) assume $f(x_n) \to 0$.

Now, $f(x_n) \geqslant b_k \cdot \dfrac{f^k(x_n)}{1+f^k(x_n)}$, for all k.

Hence $f(x_n)/b_k \geqslant f^k(x_n)\,[1 - b_k^{-1} f(x_n)]$.

Hence, $f^k(x_n) \to 0$ as $n \to \infty$.

Conversely, assume $f^k(x_n) \to 0$ for all k.

For any m, $f(x) \leqslant \sum_{k=1}^{m} b_k \cdot \dfrac{f^k(x_n)}{1+f^k(x_n)} + \sum_{k=m+1}^{\infty} a_k \to 0$ as m $\to \infty$.

(b) f is stronger than each f^k follows from (2·2) and now if f' is a *VP* stronger then f^k, then

$f'(x_n) \to 0$ implies $f^k(x_n) \to 0$ for all k and so, by (2·2) $f(x_n) \to 0$. Thus, f' is stronger than f.

(c) f is *TVP*. Then $f(x)=0$ implies $x=0$. But by definition of f, $f^k(x)=0$, for all k.

Hence $f(x) = 0$ implies $x=0$ and conversely.

Proposition 2.3—Every vectorial pseudonorm is a *VP*.

PROOF: Let f be a vectorial pseudonorm. The nontrivial part of the proof is continuity of multiplication.

Let $\{\alpha_n\}$ be a sequence of scalars with $\alpha_n \to \alpha$ and $\{x_n\}$ be a sequence of vectors with $x_n \to x$. Then,

$$f[a_n x_n - ax] = f[a_n(x_n - x) + (a_n - a)x]$$
$$\leqslant |a_n| f(x_n - x) + |a_n - a| f(x) \to 0,$$
$$(\text{as } a_n \to a \text{ and } x_n \to x, f(x_n - x) = 0).$$

Thus f is *a VP*.

3. Decomposition

To each *VP* $f: X \to R^k_+$ we associate the following subspaces of X.

$K_j(f) = \{x \in X : f_j(x) = 0,\}, \quad (j = 1, \ldots, k)$

$K(f) = \bigcap_{h=1}^k K_h(f)$

$W_j(f) = \bigcap_{h \neq j} K_h(f), \qquad (j = 1, \ldots, k)$

$$W(f) = \sum_{j=1}^{k} W_j(f).$$

Proposition 3.1—f is a *TVP* iff $K(f) = \{0\}$.

Proof : Assume, f is a *TVP*. Then, $f(x) = 0$ iff $x = 0$. Clearly, $f(x) = 0$ implies $f_j(x) = 0, \quad (j = 1, \ldots, k)$. Again, $K_j(f) = \{x \in X : f_j(x) = 0\} = \{0\}$

and $K(f) = \bigcap_{j=1}^{k} K_j(f) = \{0\}$.

Conversely, assume $f(x) = 0$ but $x \neq 0$. As $f(x) = 0, \quad f_j(x) = 0, \quad (j = 1, \ldots, k)$, hence, $x \in K_j(f) \qquad (j = 1, \ldots, k)$.

Hence, $x \in K(f) = \bigcap_{j=1}^{k} K_j(f)$, with $x \neq 0$, contrary to the hypothesis. Hence, f is a *TVP*.

Proposition 3.2 — If f is a *VP*,

$$\sum_{h \neq j} W_h(f) \subseteq K_j(f), \quad (j = 1, \ldots, k) \qquad \ldots \qquad (3\cdot1)$$

$$W(f) \subseteq W_j(f) + K_j(f), \quad (j = 1, \ldots, k) \qquad \ldots \qquad (3\cdot2)$$

Proof : The assumption $x \in \sum_{h \neq j} W_h(f)$ implies $x \notin W_j(f)$ which implies $x \in K_j(f)$.

Hence, $\sum_{h \neq j} W_h (f) \subseteq K_j (f), \quad (j = 1, .., k).$

Again, $W (f) = \sum_{j \neq 1}^{k} W_j (f) = \sum_{h \neq j}^{k} W_h (f) + W_j (f).$

$$\subseteq K_j (f) + W_j (f), \text{ (by (3·1))}.$$

Definition 3·1—A *VP* f is called regular iff $W (f) = X$.

Example—Consider $f : \mathbb{C}^3 \to R^2_+$ defined by,

$$f (x, y, z) = (| x | , | z |).$$

Then $K_1 (f) = W_2 (f) = \{ (0, y, z) : y, z \in \mathbb{C} \}$

$K_2 (f) = W_1 (f) = \{ (x, y, 0) : x, y \in \mathbb{C} \}.$

Thus, $W_1 (f) + W_2 (f) = \mathbb{C}^3$. So f is a regular *VP* and f is not a norm.

Note : Since, f defined here is a vectorial pseudonorm (Deutsch 1971) it is a *VP* by proposition (2·3).

Proposition 3.3—If $f : X \to R^k_+$ is a regular *TVP* then, $K_j (f) = \sum_{h \neq j} \oplus W_h (f)$, $(j = 1, ..., k)$.

Proof : Let $x \in K_j (f)$ then $f_j (x) = 0, \quad (j = 1, .., k)$.

Since f is regular, $W(f) = X$.

Hence $x \in X$ is of the form

$$x = \sum_{j=1}^{k} x_j \quad \text{with} \quad x_j \in W_j , \quad (j = 1, .., k).$$

Now, $f_j (x_j) = f_j (x) = 0$ implies $x_j = 0$.

Hence, $x = x_1 + .. + x_{j-1} + x_{j+1} + .. + x_k \in \sum_{h = j} W_h (f)$.

Thus, $K_j (f) \subseteq \sum_{h \neq j} W_h (f)$ and

$\sum_{h \neq j} W_h (f) \subseteq K_j (f)$, (by (3·1))

So, $K_j (f) = \sum_{h \neq j} W_h (f)$.

and $W_h, (f)$ $(h = 1,.., k)$ being independent,

$$K_j (f) = \sum_{h \neq j} \oplus W_h (f), \quad (j = 1,.., k).$$

Remark 3.1 : The converse of proposition 3·3 is not true.

Consider a *TVP* $f: \mathbb{C}^4 \to R_+^3$ given by,

$f (x_1, x_2, x_3, x_4) = [| x_1| + | x_4| , | x_2| + | x_4| , | x_3| + | x_4 |]$.

Then,

$$\begin{aligned} K_1 (f) &= \{(0, x_2, x_3, 0) : x_2, x_3 \in \mathbb{C}\} \\ K_2 (f) &= \{(x_1, 0, x_3, 0) : x_1, x_3 \in \mathbb{C}\} \\ K_3 (f) &= \{(x_1, x_2, 0, 0) : x_1, x_2 \in \mathbb{C}\} \end{aligned}$$

and

$$\begin{aligned} W_1 (f) &= \{(x_1, 0, 0, 0) : x_1 \in \mathbb{C}\} \\ W_2 (f) &= \{(0, x_2, 0, 0,) : x_2 \in \mathbb{C}\} \\ W_3 (f) &= \{(0, 0, x_3, 0) : x_3 \in \mathbb{C}\}. \end{aligned}$$

Thus,

$$K_j (f) = \sum_{h \neq j} + W_h (f), \quad (j = 1, 2,3).$$

But f is not regular, since $\Sigma W_h (f) \neq \mathbb{C}^4$.

Proposition 3.4—Let f_1 and f_2 be two *VPs* of order k on X and assume that $f_1 \leqslant f_2$ (i.e. $f_1(x) \leqslant f_2 (x)$, $\forall x \in X$).

To prove

(a) If f_1 is a *TVP* then so is f_2.

(b) If f_2 is regular, f_1 is also regular.

PROOF : If $f_1 \leqslant f_2$, $f_{1_j} \leqslant f_{2_j}$, $(j = 1,.., k)$.

Consider, $K_j (f_i) = \{x \in X : f_{i_j} (x) = 0\}$, $i = 1, 2$.

Let $x \in K_j (f_2)$, then

$f_{2_j} (x) = 0$ implies $f_{1_j} (x) = 0$, $(j = 1,.., k)$, which implies

$x \in K_j (f_1)$. Hence $K_j (f_2) \subset K_j (f_1)$.

Again,

$$W_j (f_2) = \bigcap_{h \neq j} K_h (f_2) \subseteq \bigcap_{h \neq j} K_h (f_1) = W_j (f_1).$$

Hence,

$$\sum_{j=1}^{k} W_j (f_2) \subseteq \sum_{j=1}^{k} W_j (f_1)$$

i.e.

$$W (f_2) \subseteq W (f_1) \quad \text{.. .. ..} \quad (3{\cdot}3)$$

Similarly,

$$K (f_2) \subseteq K (f_1). \quad \text{.. .. ..} \quad (3{\cdot}4)$$

(a) Now, if f_1 is total,
$K(f_1) = \{0\}$ implies $K(f_2) = \{0\}$, (by (3·4)).
Hence f_2 is also total.

(b) If f_2 is regular,
$W(f_2) = X \subseteq W(f_1)$, (by (3·2)).
Hence $W(f_1) = X$. Thus f_1 is regular.

Proposition 3.5—If f_1, f_2 are regular *TVP* of order k on X such that $f_1 \leqslant f_2$ then

$$K_j(f_1) = K_j(f_2), \qquad (j = 1, .., k) \qquad (3·5)$$

$$W_j(f_1) = W_j(f_2), \qquad (j = 1, .., k) \qquad (3·6)$$

PROOF : First we will prove, if f is a regular *TVP*,

$$X = K_j(f) \oplus W_j(f).$$

Since the *TVP* f is regular, $W(f) = X$, and as W_j is independent of K_j and $K_j(f) \cap W_j(f) = \{0\}$ the sum turns out to be direct.

Hence, we have to prove, $W(f) = K_j(f) + W_j(f)$. Clearly we have $W(f) \subseteq K_j(f) + W_j(f)$, (by (3·2)).

So for other inclusion, let, $x \in K_j(f) + W_j(f)$,
then, $x = x_1 + x_2$ with $x_1 \in K_j(f)$, $x_2 \in W_j(f)$,
i.e. $x_1 \in K_j(f)$, $x_2 \in W_j(f)$ and $x_1 \in W_h(f)$, $h \neq j$.

Thus, $$x \in \sum_{h=1}^{k} W_h(f) = W(f).$$

i.e. $$W(f) = K_j(f) \oplus W_j(f) = X.$$

Now, f_1, f_2 being regular,

$$X = K_j(f_1) \oplus W_j(f_1) = K_j(f_2) \oplus W_j(f_2),$$ and on account of dimension argument, we get,

$$K_j(f_1) = K_j(f_2) \text{ and } W_j(f_1), = W_j(f_2) \quad (j = 1, .., k).$$

4. THE DUAL OF a TVP

Let $f : X \to R_+^k$ be a given TVP. Consider the mapping $h_j : X \to R$ defined by,

$$h_j(y) = \sup_{x \in W_j(f),\, x \neq 0} \frac{|(y, x)|}{f_j(x)}, \quad y \in X.$$

for each $j = 1,..,k$. Then the mapping $f^D : X \longrightarrow R_+^k$ defined by,

$$f^D\ (y) = [h_1\ (y), \quad , h_k\ (y)] \quad , \quad y \in X$$

is a *VP* of order k on X.

(Since each h_j is a paranorm which is not total, f^D is not total). This mapping f^D is called the dual of f.

Proposition 4.1—If f is a *TVP*, then

(a) $K_j\ (f^D) = [W_j\ (f)]^{\perp}$, $(j = 1,..,k)$.

(b) $W_j\ (f^D) \supseteq [K_j\ (f)]^{\perp}$, $(j = 1,..,k)$.

(c) $K\ (f^D) = [W\ (f)]^{\perp}$, and

(*d*) f^D is regular.

Proof : (a) $K_j\ (f^D) = \{ y \in X \ : f_j^D\ (y) = 0 \}, \quad (j = 1..,k)$

$$= \{ y \in X \ : h_j\ (y) = 0 \}, \text{ (by definition of } f_j^D)$$

$$= \{ y \in X \ : (y, X) = 0 \} = [W_j\ (f)]^{\perp}.$$

(b) $W_j\ (f^D) = \bigcap_{h \neq j} K_h\ (f^D) = \bigcap_{h \neq j} [W_h\ (f)]^{\perp}$

$$= \sum_{h \neq j} [W_h\ (f)]^{\perp} \supseteq [K_j\ (f)]^{\perp} \text{ (by (3·1)).}$$

(c) $K\ (f^D) = \bigcap_{h=1}^{k} K_h\ (f^D) = \bigcap_{h=1}^{k} [W_h\ (f)^{\perp} = [\sum_{1}^{k} W_h\ (f)]^{\perp} = [W\ (f)]^{\perp}.$

(d) $W\ (f^D) = \sum_{h=1}^{k} W_h\ (f^D) \supseteq \sum_{j=1}^{k} [K_j\ (f)]^{\perp} = [\bigcap_{h=1}^{k} K_h\ (f)]^{\perp} = X.$

Hence f^D is a regular VP.

Corollary—f^D is a TVP iff f is regular. Here $X = \mathbb{C}^n$.

Proof : If f is regular, $W\ (f) = \mathbb{C}^n$.

Hence $[W\ (f)]^{\perp} = (\mathbb{C}^n)^{\perp} = \{ 0 \}.$

But from (c) above, $K\ (f^D) = \{0\}$. Thus f^D is a TVP.

If f^D is a TVP, $K\ (f^D) = \{ 0 \}$, then by (c)

$[W\ (f)]^{\perp} = \{ 0 \}$. Hence $[W\ (f)]^{\perp\perp} = \{ 0 \}^{\perp} = \mathbb{C}^n$.

What remains to show is that $W\ (f)$ is closed. For that recall that, "If S is a non-empty subset of a Hilbert space H, then $S^{\perp\perp}$ is the closure of the set of all linear combinations of vectors of S. i.e. S is the smallest closed linear subspace of H". (Simmons 1963). The proof is now complete.

5. Remarks

At the end it is remarked that the following statements could also be proved as per arguments used in section 2.

(1) If $f(x - y) = 0$ then $f(x) = f(y)$. The converse may not hold.

(2) If $f(x) = 0$, then $f(x - y) = f(y)$, for all $x \in X$.

(3) If f is a TVP and g is stronger than f, then g is also a TVP

(4) If f is a TVP, g is a VP, then $f+g$ is also a TVP and $f + g$ is stronger than g.

We observe that all the results established here also hold for vectorial norms (Deutsch 1971). The special case $k = 1$ also gives the results for the paranorms (Wilansky 1964).

References

Deutsch, Emeric (1971). On vectorial norms and pseudonorms. *Proc. Am. math. Soc.*, **28**, 18-24

Fielder, M. and Ptak V. (1972). Generalized norms of matrices and the location of the spectrum, *Czech. Math. J.*, **12**, (87), 558-71. *MR* 4187.

Kantorovich, L. V. (1939). The method of successive approximations for functional equations, *Acta Math.*, **71**, 63-97. *MR* 1, 18.

Robert, F. (1964). Norms vectorilles de vecteurs et de matrices, *Rev. Franc. Traitement Inform. Chiffres,* **7**, 261-99. *MR* 34 5261.

Robert, F. (1965). Sur les normes vectorilles regulieres sur un espace vectorial de dimension finie. *C.R. Acad. Sci. Paris* **261**, 5173-76. *MR* 33 543.

Simmons, G. (1963). Introduction to Topology and Modern Analysis. Mc Graw-Hill Book Co., Inc., New York.

Stoer J. (1968). Lower bounds of matrices. *Number Math.*, **12**, 146, 158. *MR* 38 5378.

Wilansky, Albert (1964). Functional Analysis, Blaisdell, New York.

Yosida, K. (1965). Functional Analysis, Springer-Verlag, Berlin.