GENERALIZED BRA AND KET VECTORS IN INFINITESIMAL GEOMETRY

by N. N. Ghosh, *Indian Association for the Cultivation of Science, Calcutta 32*

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This paper gives a more comprehensive treatment of bra and ket vectors than what was discussed in a previous paper (Ghosh 1973). Starting from a two-fold vector the author has introduced here higher order k-fold bra and ket vectors with their matrix representations involving k-fold matrix units and elements expressed as functional determinants.

§ 1. Let us start with the general infinitesimal vector of the second order (Ghosh 1973)

$$\mu\,(x_r\; x_s)\sqrt{dx_r}\;\sqrt{dx_s},\;\; r, s = 1, 2, \ldots\ldots \infty \tag{1.1}$$

expressed in the form of an infinite matrix.
Associated with the above there are an infinite number of ket vectors of the first order with components

$$M_p^h = \mu\,(x_p\; x_h)\,\sqrt{dx_p},\quad h = 1, 2, \ldots\ldots \infty. \tag{1.2}$$

p running over the values $1, 2 \ldots\ldots$, ($x_{k-\infty}$ being finite).
The components of the bra vectors of the first order are written as

$$M_h^p = \mu\,(x_h\; x_p)\sqrt{dx_p},\; h = 1, 2, \ldots \ldots \infty \tag{1.3}$$

The index p in the above is called an effective index while index h has fixed values.

Using the symbol μ_p^h to denote $\mu\,(x_p\;\; x_h)$ let us take a pair of ket vectors M^i, M^j with components written in the form of an array

$$\begin{vmatrix} \mu_1^i \sqrt{dx_1} & \mu_2^i \sqrt{dx_2} & \cdots \mu_p^i \sqrt{dx_p} & \cdots & \infty \\ \mu_1^j \sqrt{dx} & \mu_2^j \sqrt{dx_2} & \ldots\; \mu_p^j \sqrt{dx_p} & \cdots & \infty \end{vmatrix}. \tag{1.4}$$

We now construct a two-fold ket vector M^{ij} of the second order with two-fold infinity of components in determinantal form

$$M_{pq}^{ij} = \begin{vmatrix} \mu_p^i & \mu_q^i \\ \mu_p^j & \mu_q^j \end{vmatrix} \sqrt{dx_p}\;\; \sqrt{dx_q}\;, p < q = 1, 2 \ldots \infty \tag{1.5}$$

containing a pair of effective indices (p, q) running over an ascending set of two-fold numbers

$$(12)\,(13)\quad(23)\,(14)\quad(24)\,(34)\quad(15)\,(25)\quad(35)\,(45)\,\ldots$$
$$(1k)\,(2k)\,\ldots\,(k-1, k)\,\ldots\,k \to \infty. \tag{1.6}$$

It is to be noted that the numbers in (1.6) comprized within a given value of k is k_{e_2}. Remembering that the square of the array (1.4) is equal to the square of the norm of the two-fold vector M^{ij} we get

$$\left\{ \begin{matrix} M^{ij} \\ M^{ij} \end{matrix} \right\} = \left\{ \begin{matrix} M^i\; M^j \\ M^i\; M^j \end{matrix} \right\} \tag{1.7}$$

If K^{kl} denote the two-fold ket vector corresponding to another infinitesimal vector of the second order with components

$$\nu\,(x_r\, x_s) \sqrt{dx_r}\; \sqrt{dx_s}\;,\; r, s = 1, 2 \cdots \infty \tag{1.8}$$

we have

$$\left\{ \begin{matrix} M^{ij} \\ N^{kl} \end{matrix} \right\} = \left\{ \begin{matrix} M^i\; M^j \\ N^k\; N^l \end{matrix} \right\}. \tag{1.9}$$

With a pair of bra vectors $M_i\, M_j$ in (1.3) we can construct a two-fold bra vector M_{ij} with components

$$M^{pq}_{ij} = \begin{vmatrix} \mu^p_i & \mu^q_i \\ \mu^p_j & \mu^q_j \end{vmatrix} \sqrt{dx_p}\;\; \sqrt{dx_q}\;,\; p < q = 1, 2, \ldots \infty \tag{1.10}$$

obtaining relations similar to (1.7), (1.9).

It must be noted in (1.5) or (1.10) that if we give a fixed value to p or q the resulting determinants will represent a partial set of components of M^{ij}, or M_{ij}. We observe that the column and the row vectors (1 ; 5, 10) generate a general two-fold matrix M^{rs}_{pq} with row numbers and column numbers running over the two-fold infinity of values (1.6).

2. Matrix Representation of Vectors

Corresponding to the first set (1, 2, 3) we form the couple of matrices (Ghosh 1940)

$$M^i = \sum_{p=1}^{\infty} \mu^i_p \sqrt{dx_p}\,(e_{p+1,\,1} + e_{1,\,p+1}) \tag{2.1}$$

$$\tilde{M}^i = \sum_{p=1}^{\infty} \mu^p_i \sqrt{dx_p}\,(e_{p+1,1} + e_{1,p+1}), \tag{2.2}$$

where e's denote matrix units.

In the above representation the non-zero elements of the vector appear only in the first row and the first column, all other elements being zero.

Introducing the unit matrix (Ghosh 1973) expressd shortly as.

$$U = \sum_{p=1}^{\infty} e_{p+1,p+1} \tag{2.3}$$

we can write (2.1) and (2.2)

$$M^i = UM^i + M^i U$$

$$\tilde{M}^i = U\tilde{M} + \tilde{M}^i U \tag{2.4}$$

forming the product of a pair of matrices M^i, N^j we get

$$M^i N^j = UM^i N^j U + M^i U N^j \tag{2.5}$$

where $UM^i N^j U$ represents the infinitesimal vector of the second order in matrrix form

$$\sum_{p,q=1}^{\infty} \mu_p^i \nu_q^j \sqrt{dx_p} \sqrt{dx_q} \; e_{p+1,q+1} \tag{2.6}$$

and, $M^i U N^j$ denotes

$$\sum_{p=1}^{\infty} \mu_p^i \; \nu_p^j \; dx_p \; e_{11} \begin{Bmatrix} M^i \\ N^j \end{Bmatrix} e_{11}. \tag{2.7}$$

Product of three vectors M^i, N^j P^k may be expressed as

$$M^i N^j P^k = UM^i \begin{Bmatrix} N^j \\ P^k \end{Bmatrix} + \begin{Bmatrix} M^i \\ N^j \end{Bmatrix} P^k U. \tag{2.8}$$

It follows from the above that a matrix M^i satisfies the characteristic equation

$$(M^i)^3 = \begin{Bmatrix} M^i \\ M^i \end{Bmatrix} M^i. \tag{2.9}$$

To transform an infinitesimal vector M^i we make use of the transformation matrix

$$\phi = \sum_{p,q=1}^{\infty} \phi\,(x_p \; x_q) \sqrt{dx_p} \sqrt{dx_q} \,(e_{p+1\,,\,q+1}) \tag{2.10}$$

and write

$$M'^i = \phi M^i + M^i_{\cdot} \tilde{\phi} \tag{2.11}$$

Where $\tilde{\phi}$ denotes the transposed of ϕ.

A matrix algebra along these lines has been framed in an earlier paper (Ghosh 1940) with applications to rigid body motion. Next we proceed to the matrix representation of a two-fold Ket vector M^{ij} defined in (1.5). If (pq) and $(p'q')$ denote a pair of consecutive members in (1.6) then following (2.1) M^{ij} in matrix form becomes

$$M^{ij} = \sum_{(pq)}^{\infty} \mu_{pq}^{ij} \sqrt{dx_p} \sqrt{dx_q} \left(e_{p'q',\,12} + e_{12,\,p'q'}\right) \tag{2.12}$$

where the summation (pq) ranges over the values (12), (13), (23) (14)... in succession.

Introducing the diagonal unit matrix

$$U = \sum_{(pq)}^{\infty} e_{pq,\,pq} \tag{2.13}$$

where the first element $e_{12,12}$ is zero we can write (2.12) in the form

$$M^{ij} = UM^{ij} + M^{ij}\,U \tag{2.14}$$

where $UM^{ij}\,U = 0.$

The product of two vectors can be written as

$$M^{ij}\,N^{kl} = UM^{ij}\,N^{kl}U + M^{ij}\,UN^{kl} \tag{2.15}$$

where

$$M^{ij}\,UN^{kl} = \begin{Bmatrix} M^{ij} \\ N^{kl} \end{Bmatrix} e_{12,12} \tag{2.16}$$

and $UM^{ij}\,N^{kl}\,U$ represents

$$\sum_{(pq)\,(rs)}^{\infty} \mu_{pq}^{ij}\,\nu_{rs}^{kl} \sqrt{dx_p\,dx_q}\,\sqrt{dx_r\,dx_s}\; e_{p'q',r's'} \tag{2.17}$$

a simple two-fold matrix in general form.

Product of three matrices M^{ij}, N^{kl}, P^{mn} may be expressed as

$$UM^{ij} \begin{Bmatrix} N^{kl} \\ P^{mn} \end{Bmatrix} + \begin{Bmatrix} M^{ij} \\ N^{kl} \end{Bmatrix} P^{mn}\,U. \tag{2.18}$$

Generally the product of an even number of matrices is of type (2.15) and for an odd number the product is of type (2.18). From (2.18) it follows that M^{ij} satisfies the characteristic equation

$$\left(M^{ij}\right)^3 = \begin{Bmatrix} M^{ij} \\ M^{ij} \end{Bmatrix} M^{ij}. \tag{2.19}$$

Let us consider the simple ante-symmetric two-fold operator

$$\alpha^{ij} = U\left(M^{ij}\,N^{ij} - N^{ij}\,M^{ij}\right) \tag{2.20}$$

Operating upon P^{kl} we get

$$P'^{kl} = M^{ij} \begin{Bmatrix} N^{ij} \\ P^{kl} \end{Bmatrix} - \begin{Bmatrix} M^{ij} \\ P^{kl} \end{Bmatrix} N^{ij}\;. \tag{2.21}$$

P^{kl} = is transformed into a vector lying in the plane of M^{ij}, N^{ij} orthogonal to P^{kl}. It may be seen that α^{ij} satisfies the characteristic equation

$$(\alpha^{ij})^3 = -\begin{Bmatrix} M^{ij} N^{ij} \\ M^{ij} N^{ij} \end{Bmatrix} \alpha^{ij} \tag{2.22}$$

For a two-fold bra vector similar treatment may be adopted. A general two-fold matrix is expressible in the form

$$\sum_{pq,\ rs}^{\infty} \mu_{pq}^{rs} \sqrt{dx_p\, dx_q}\ \sqrt{dx_r\, dx_s}\ e_{p'q',r's'}\,. \tag{2.23}$$

The above treatment can be easily extended to tackle k-fold infinitesimal vectors (Ghosh 1970). For the matrix representation of these vectors we have to introduce k-fold matrix units involving several indices having elements expressed as functional determinants of kth order. For various operations among these matrices we shall have recourse to determinantal identities obtained in a previous paper (Ghosh 1973).

References

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