NOTE ON THE CHARACTER QUASIGROUP OF AN ABELIAN TOPOLOGICAL QUASIGROUP

by PHULLENDU DAS, *Department of Mathematics, Visva-Bharati University, P. O. Santiniketan, West Bengal*

(Communicated by F. C. Auluck, F. N. A.)

(*Received 8 March 1973 ; after revision 28 March 1974*)

The construction of a character group plays an important role in the study of locally compact commutative topological groups. The object of the present note is to propose a definition of the character quasigroup X of a locally compact Abelian topological quasigroup G having an idempotent element and then to establish a result by using the notion of principal isotopy with the help of which the relations between G and X can be readily obtained from the corresponding relations between a locally compact commutative topological group and its character group.

INTRODUCTION

A quasigroup G is a system of three compositions, viz. product (.), right division (/) and left division (\ ) such that $a.\, b = c \Longleftrightarrow c/b = a \Longleftrightarrow c \backslash a = b$ for every $a, b, c \in G$ (Bruck 1958). If for every $a, b, c, d \in G$, $(a.\, b).\ (c.\, d) = (a.\, c).\ (b.\, d)$, then G is said to be an Abelian quasigroup (Murdoch 1939).

A topological quasigroup (G, τ) is a quasigroup G endowed with a topology τ w. r. t. which the operations of G are continuous. τ is said to be a quasigroup topology in G. If G be an Abelian quasigroup, then (G, τ) is said to be an Abelian topological quasigroup (Das 1968).

The construction of a character group plays an important role in the study of locally compact commutative topological groups. The object of the present note is to propose a definition of the character quasigroup X of a locally compact Abelian topological quasigroup G having an idempotent element and then to establish a result by using the notion of principal isotopy with the help of which the relations between G and X can be readily obtained from the corresponding relations between a locally compact commutative topological group and its character group.

CHARACTER QUASIGROUP

Let K be the one-dimesional circle group. If the addition in K be denoted by $+$ and if $\oplus$ be defined by the relation

$$a \oplus b = -a - b = (a)\, i + (b)\, i \text{ for every } a, b \in K \tag{1}$$

where $i : K \to K$ is defined by the relation $(a)\, i = -a$ for every $a \in K$, then $(K, \oplus)$ is a locally compact Abelian topological quasigroup having the null element 0 in $(K, +)$ as an idempotent element and satisfying the second axiom of countability. Also we note that (i, i, I_K) is a principal isotopy of $(K, \oplus)$ onto $(K, +)$. [A principal isotopy of a topological groupoid $(G, .)$ onto a topological groupoid (G, o) is a triple (α, β, I_G) where α, β are homeomorphisms of G and I_G is the identity mapping of G such that $x.y = (x)\alpha \; o \; (y)\beta$ (Das 1967).]

Now let $(G, \oplus)$ be a locally compact Abelian (additive) topological quasigroup having an idempotent element p. A homomorphism of $(G, \oplus)$ onto $(K, \oplus)$ will be called a character of $(G, \oplus)$. Let X be the set of all characters of $(G, \oplus)$. If $\alpha, \beta \in X$, we define $\alpha \oplus \beta$ by the relation

$$(x)(\alpha \oplus \beta) = (x)\alpha \oplus (x)\beta \text{ for every } x \in G \tag{2}$$

Then $\alpha \oplus \beta \in X$. Also $(X, \oplus)$ is an Abelian quasigroup having θ defined by the relation $(x)\theta = 0$ for every $x \in G$ as an idempotent element.

We note the following result (Das 1968),

Lemma 1 :—If τ be a quasigroup topology in a quasigroup G having an idempotent element p and if $(p)\tau = \mho$, then

(*a*) $\overleftarrow{p} \leqslant \mho$;

(*b*) $\mho . \mho \leqslant \mho$, $\mho / \mho \leqslant \mho$, $\mho \backslash \mho \leqslant \mho$;

(*c*) if $b \in U . (a \backslash p)$ [or $b \in (a/p)\, U$] where $a, b \in G$ and $U \in \mho$, then there exists a $V \in \mho$ such that

$$V . (b \backslash p) \subset U . (a \backslash p) \text{ [or } (b/p) . V \subset (a/p) . U \text{].}$$

Conversely if $\mho$ be a filter in an Abelian quasigroup G having an idempotent element p satisfying (*a*), (*b*), (*c*) then there exists a quasigroup topology τ in G such that $(p)\tau = \mho$.

Now let $\mho$ be the neighbourhoods filter of O in $(K, \oplus)$ and $C(G)$ the family of all compact subsets of G. If $F \in C(G)$ and $U \in \mho$, let

$$\langle F, U \rangle = \{\alpha \in X ; (F)\alpha \subset U\}.$$

Let $$U' = \{\langle F, \mho' \rangle ; F \in C(G), U \in \mho\}.$$

Then it can be readily verified that $\mho'$ is a filter in $(X \oplus)$ satisfying the conditions (*a*), (*b*), (*c*) of Lemma 1.

$\therefore$ There exists a quasigroup topology τ' in $(X, \oplus)$ such that $(\theta)\, \tau' = \mho''$.

Definition—The topological quasigroup $(X, \tau', \oplus)$ is defined to be the character quasigroup of $(G, \oplus)$.

Let $(G, \oplus)$ be a locally compact Abelian topological quasigroup having an idempotent element p. A new composition $+$ is defined in G by the relation

$$a + b = a/p \oplus b \setminus p = (a)R_p^{-1} \oplus (b)L_p^{-1} \quad \text{for every } a, b \in G. \tag{3}$$

Since R_p, L_p are homeomorphisms (Das 1968), $(R_p^{-1}, L_p^{-1}, I_G)$ is a principal isotopy of $(G, +)$ onto $(G, \oplus)$. Since an isotope of a topological quasigroup is a topological quasigroup (Das 1967), $(G, +)$ is a topological quasigroup. Also $(G, +)$ is Abelian and has p as its null element. Since an Abelian quasigroup having the identity element is a commutative group (Murdoch 1939) and a topological quasigroup in which the associative property holds is a topological group (Das 1968), it follows that $(G, +)$ is a locally compact commutative topological group.

Now let $(X, \oplus)$ and $X', +)$ be the character quasigroup and character group of $(G, \oplus)$ and $(G, +)$ respectivaly. Then $X = X'$. For, let $\alpha \in X$.

Then $(p)\alpha = (p \oplus p)\alpha = (p)\alpha \oplus (p)\alpha = -(p)\alpha - (p)\alpha$ gives $3.\ (p)\alpha = 0$.
$\therefore\ (p)\alpha = 0$.

$$\begin{aligned} \therefore (a+b)\alpha &= (a/p \oplus b \setminus p)\alpha \\ &= (a/p)\alpha \oplus (b \setminus p)\,\alpha \\ &= -(a)\alpha \oplus -(b)\alpha \quad [\because (p)\alpha = 0] \\ &= (a)\alpha + (b)\alpha \end{aligned}$$

$\therefore\ \alpha \in X'$. $\therefore\ X \subset X'$.
Similarly $X' \subset X$. $\therefore\ X = X'$.

Now let $\alpha, \beta \in X$. Then for every $x \in G$,

$$\begin{aligned} (x)\,[\,\alpha \oplus \beta)\,] &= (x)\,\alpha \oplus (x)\,\beta \\ &= -(x)\,\alpha - (x)\,\beta = (x)\,[\,(\alpha)i + (\beta)i\,] \end{aligned}$$

$$\therefore\ \alpha \oplus \beta = (\alpha)i + (\beta)i \quad \text{for every } \alpha, \beta, \in X \tag{4}$$

$\therefore$ (i, i, I_X) is a principal isotopy of $(X, \oplus)$ onto $(X, +)$.

Thus from the above discussion we can state as follows :

Theorem 1—Every locally compact Abelian topological quasigroup $(G, \oplus)$ having an idempotent element has a principal isotope which is a locally compact commutative topological group $(G, +)$ such that (i, i, I_X) is a principal isotopy of the character quasigroup $(X, \oplus)$ of $(G, \oplus)$ onto the character group $(X, +)$ of $(G, +)$.

Thus to obtain the character quasigroup of a locally compact Abelian topological quasigroup $(G, \oplus)$ having an idempotent element p we can proceed as follows : Firstly, we find a locally compact commutative topological group $(G, +)$ such that $(R_p^{-1}, L_p^{-1}, I_G)$ is a principal isotopy of $(G, +)$ onto $(G, \oplus)$. Next we find the character group $(X, +)$ of $(G, +)$. Finally we find $(X, \oplus)$ such that (i, i, I_X) is a principal isotopy of $(X, \oplus)$ onto $(X, +)$. Then $(X, \oplus)$ is the character quasigroup of $(G, \oplus)$.

Theorem 2—If $(X, \oplus)$ be the character quasigroup of a locally compact Abelian Hausdorff topological quasigroup $(G, \oplus)$ having an idempotent element, then $(G, \oplus)$ is the character quasigroup of $(X, \oplus)$.

Proof: With the same notations as before, $(X, +)$ is the locally compact commutative Hausdorff topological group such that $(R_\theta^{-1}, L_\theta^{-1}, I_X)$ is a principal isotopy ot $(X, +)$ onto $(X, \oplus)$. Now by Pontrjagin duality theorem, $(G, +)$ is the character group of $(X, +)$. Since $(G, \oplus)$ is the quasigroup such that (i, i, I_G) is a principal isotopy of $(G, \oplus)$ onto $(G, +)$, $(G, \oplus)$ is the character quasigroup of $(X, \oplus)$.

Using the same method and the corresponding results for topological groups, the following two results can be established.

Theorem 3—Let $(X, \oplus)$ be the character quasigroup of a locally compact Abelian Hausdorff topological quasigroup $(G, \oplus)$ having an idempotent element. If G is compact, then X is descrete and if G is discrete, then X is compact.

Theorem 4—If X_i be the character quasigroup of the locally compact Abelian Hausdorff topological quasigroup G_i having an idempotent element for $i=1, 2, \ldots, n$ and if X and G are the direct sums of the quasigroups X_1, X_2, $-X_n$ and $G_1, G_2 \ldots \ldots G_n$ respectively, then X is the character quasigroup of G.

In fact, using the above principle a relation between a locally compact Abelian topological quasigroup having an idempotent element and its character quasigroup can be obtained from any relation between a locally compact commutative topological group and its character group.

REFERENCES

Bruck, R, H. (1958). A survey of binary systems. Springer Verlag,

Das, Phullendu (1967). A note on homotopy and isotopy of topological groupoids. *Prog. Math.*, **1**, 21-28,

—— (1968). Topological quasigroups. D. Phill thesis submitted to University of Calcutta.

Husain, T. (1966). Introduction to Topological Groups. W. B. Saunders' Company.

Murdoch, D. C. (1939). Quasigroups which satisfy certain generalised associative laws. *Am. J. Math.*, **61**, 509-22.

Pontrjagin, L. (1958). Topological Groups. Princeton University Press.