A MATHEMATICAL MODEL FOR IMBIBITION IN DOUBLE PHASE FLOW THROUGH POROUS MEDIUM WITH CAPILLARY PRESSURE

by S. K. Mishra, S. P. Verma *and* Dilip C. Shah,
Department of Applied Mathematics, S. V. Regional College of Engineering and Technology, Surat 395001.

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The present paper analytically discusses a theoretical model for imbibition phenomena of ground water replenishment in double phase ffow through homogeneous porous medium with capillary pressure. The basic assumptions underlying the present investigation are that the 'oil' and 'water' form two immiscible liquid phases and the later represents preferentially wetting phase. A mathematical formulation leads to a nonlinear differential equation which has been reduced to the well-known Abel's equation of second kind by using a similarity technique. A formal mathematical solution of the later has been obtained in terms of transcendental functions.

1. Introduction

It is well known that if a porous medium filled with some fluid is brought into contact with another fluid, which preferentially wets the medium, then there is a spontaneous flow of the wetting fluid into the medium and a counter flow of the resident fluid from the medium. This phenomenon is called imbibition (Scheidegger 1960) and been discussed by Graham and Richardson (1959), Rijik (1960), and Verma (1969, 1970, 1971, 1972) from different point of view.

In this paper we have obtained a mathematical solution of the non-linear differential system governing imbibition phenomenon in the investigated model which has been reduced to the well known Abel's equation of the second kind by using similarity technique (Hansen 1964). A formal mathematical solution of the problem has been obtained in terms of transcendental function.

Mathematical Formulation of the Problem

We consider here that a finite cylindrical piece of homogeneous porous matrix containing viscous oil, is completely surrounded by an impermeable surface except for one end of the cylinder which is labelled as the imbibition phase and this end is exposed to an adjacent formation of 'injected' water. It is assumed that the injected water and the viscous oil are two immiscible liquids of different salinities with small viscosity difference ; the former represents the preferentially wetting and less viscous phase.

The main interest of the present investigation is to obtain an analytical solution of the nonlinear differential equation of imbibition under the special conditions of our problem.

The seepage velocity of oil (V_0) and water (V_w) from Darcy's law are written as ;

$$V_o = -\frac{K_o}{\mu_o} K \frac{\partial p_o}{\partial x}, \tag{2.1}$$

$$V_w = -\frac{K_w}{\mu_w} K \frac{\partial p_w}{\partial x}, \tag{2.2}$$

where K is the permeability of the medium, K_0 and K_w are the relative permeabilities of oil and water respectively, which are the functions of S_o and S_w (the saturation of oil and water respectively) ; p_0 and p_w are the pressures of oil and water, and μ_0 and μ_w are the viscosities of oil and water.

The equation of continuity for water can be written as

$$\varphi \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{2.3}$$

where φ is the porosity of the medium.

The analytical condition (Scheidegger 1960) governing imbibition phenomenon is

$$V_o = -V_w. \tag{2.4}$$

From the definition of capillary pressure (p_c) as the pressure discontinuity between the flowing phases yields.

$$p_c = p_o - p_w. \tag{2.5}$$

Combining eqns. (2.1), (2.2), (2.4) and (2.5) we get

$$\frac{\partial p_w}{\partial x} = -\left\{\frac{K_o / \mu_o}{(K_o / \mu_o + K_w / \mu_o}\right\} \frac{\partial p_c}{\partial x}. \tag{2.6}$$

Substituting the above in eqn. (2.2), we have

$$V_w = K\left\{\frac{K_0 / \mu_0 \,.\, K_w / \mu_w}{(K_o / \mu_o + K_w / \mu_w)}\right\}\frac{\partial p_c}{\partial x}\,. \tag{2.7}$$

Equation (2.3) and (2.7) yield

$$\varphi\,\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[K\frac{K_o / \mu_o \,.\, K_w / \mu_w}{(K_o / \mu_o + K_w / \mu_w)}\cdot\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x}\right] = 0. \tag{2.8}$$

This is the desired differential equation describing the imbibition phenomenon whose approximate mathematical solution has been obtained below.

Similarity Solution

Since the present investigation involves water and a viscous oil, therefore according to Scheidegger (1960) approximation, we may write eqn. (2.8.) in the form

$$\varphi\,\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[K\frac{K_o}{\mu_o}\cdot\frac{dp_c}{dS_w}\cdot\frac{\partial S_w}{\partial x}\right] = 0 \tag{3.1}$$

$$\text{as}\;\frac{K_o / \mu_o \,.\, K_w / \mu_w}{(K_o / \mu_o + K_w / \mu_w)} \approx K_o/\mu_o\,.$$

At this stage for definiteness of the mathematical analysis, we assume a standard form of Jones (1961) and Muskat (1949) as

$$K_w = S_w^3,\; K_o = 1 - \alpha S_w,\; \alpha = 1.11$$

and

$$p_c = \beta S_w. \tag{3.2}$$

Substituting the values from eqn. (3.2) into (3.1) we get

$$\varphi\,\frac{\partial S_w}{\partial t} + \frac{K}{\mu_o}\beta\,\frac{\partial}{\partial x}\left[(1-\alpha S_w)\,\frac{\partial S_w}{\partial x}\right] \equiv 0 \tag{3.3}$$

We now choose new variables to convert equation (3.3) into a dimensionless form

$$X = \frac{x}{L}\,,\; T = \frac{K\beta}{\varphi\,\mu_o\,L^2}\,t\,,\;\text{and}\;\; \overset{*}{S_w} = (1-\alpha\, S_w)$$

and then eqn. (3.3) takes form

$$\frac{\partial \overset{*}{S_w}}{\partial T} + \overset{*}{S_w}\frac{\partial^2 \overset{*}{S_w}}{\partial X^2} + \left(\frac{\partial \overset{*}{S_w}}{\partial X}\right)^2 = 0. \tag{3.4}$$

To solve eqn. (3.4), we use Birkhof's technique of one parameter group transformations (Hansen 1964). Let a group T_1 consisting of a set of transformations defined as

$$T_1: \quad \bar{X} = a^q X, \bar{T} = a^r T, \quad \text{and} \quad \bar{S}_w^* = a^s S_w^* \tag{3.5}$$

where the parameter $a \neq 0$, and q, r, s are real numbers to be determined. Substituting eqn (3.5) into eqn. (3.4) we obtain

$$a^{r-s} \frac{\partial \bar{S}_w^*}{\partial \bar{T}} + a^{2q-2s} \bar{S}_w^* \frac{\partial^2 \bar{S}_w^*}{\partial \bar{X}^2} + a^{2q-2s} \left(\frac{\partial \bar{S}_w^*}{\partial \bar{X}} \right)^2 = 0 . \tag{3.6}$$

Equation (3.6) is absolute conformally invariant under T_1, provided

$$2q - 2s = r - s. \tag{3.7}$$

Now, we choose to eliminate T so that the solution of eqn. (3.7) for $r \neq 0$ an equivalnet to the solution of

$$2\frac{q}{r} - 2\frac{s}{r} = 1 - \frac{s}{r} . \tag{3.8}$$

Choosing an arbitrary constant A and then setting

$$\frac{s}{r} = A, \tag{3.9}$$

combining eqn. (3.9) with (3.8) we get

$$\frac{q}{r} = \frac{1+A}{2} \quad B/2$$

where B is a constant.

Thus the invariants of the group T_1 are given by

$$\eta = X/T^{B/2}, \; F(\eta) = S_w^* (X,T)/T^A . \tag{3.10}$$

Then eqn. (3.4) in terms of the variables η and $F(\eta)$ is

$$T^{A-1} \left[AF(\eta) - \frac{B}{2} \eta F'(\eta) + F'^2(\eta) + F(\eta) F''(\eta) \right] = 0. \tag{3.11}$$

Since $\quad T^{A-1} \neq 0,$

therefore

$$FF'' + F'^2 - \frac{B}{2} \eta F' + AF = 0. \tag{3.12}$$

This a nonlinear ordinary differential equation of second order.

The equation (3.12) can be solved by considering the substitution

$$F = \eta^2 u(z) ; \quad Z = \log \eta$$

and $\quad u'(z) = p. \tag{3.13}$

Then eqn. (3.12) takes the form

$$upp'(u) + p^2 + \left(7u - \frac{B}{2}\right) p + (6u-1)\, u = 0, \tag{3.14}$$

This is the Abel's equation of second kind (Murphy 1969), whose solution can be obtained by considering a new substitution

$$up = \log v(z). \tag{3.15}$$

Substituting eqn. (3.15) into eqn. (3.14), we get

$$\frac{1}{V}\frac{dV}{du} = \frac{B}{2} - 7u + (1-6u)\frac{u}{p}. \tag{3.16}$$

Further if we suppose $\log V = M(Z)$ then eqn. (3.16) can take the form

$$\frac{dM}{dZ} + \left(7u - \frac{B}{2}\right)\frac{M}{u} + (6u-1)\, u = 0. \tag{3.17}$$

The solution of eqn. (3.17) is

$$Me^{\int\left(7 - \frac{B}{2u}\right) dz} = \int u\,(1-6u)\, dz\, e^{\int\left(7 - \frac{B}{2u}\right) dz} + C \tag{3.18}$$

where C is a constant of integration.

$$Me^{\int\left(\frac{7}{\eta} - \frac{B}{2F}\eta\right) d\eta} = \int \frac{F}{\eta^3}\left(1 - \frac{6F}{\eta^2}\right) d\eta\, e^{\int\left(\frac{7}{\eta} - \frac{B}{2F}\eta\right) d\eta} + C. \tag{3.19}$$

The value of C is determined by considering an appropriate boundary condition when $t=0$.

Equation (3.19) gives the formal solution in terms of transcendental functions.

Conclusion

In this paper we have obtained the analytical solution of imbibition phenomenon by using a similarity technique of one parameter group transformation.. The mathematical formulation leads to a nonlinear differential equation which has been reduced to Abel's equation of second kind by using a similarity solution and its formal solution is obtained in terms of transcendental functions. We have not included any numerical illustration or graphical representations due to our particular interest in the analytical solution but the same can be easily obtained by using field values of the characteristics parameters.

Notwithstanding the limitations of the present analysis it is believed that the present similarity solution will provide useful theoretical informations of at least one complicated case of imbibition phenomenon of two immiscible liquid flow problems.

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