COUPLED TORSIONAL AND LONGITUDINAL VIBRATIONS OF A PRETWISTED SLENDER BEAM UNDER AERODYNAMIC COUPLINGS IN A CENTRIFUGAL FORCE FIELD

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The differential equations for the coupled torsional and longitudinal vibrations of a pretwisted slender beam under aerodynamic couplings in a centrifugal force field are obtained. A method based on series solution is used to obtain the critical speed of the steady flow and the corresponding fundamental frequency of vibration.

INTRODUCTION

The analysis presented in this paper considers the longitudinal vibrations of a pretwisted slender beam that could represent a turbine blade of simple geometry. The beam is attached to a disc of radius r_0 and the disc rotates with a constant angular velocity Ω (Fig. 1). It is well known that when a thin bar is under torsion there is a slight decrease in distance between cross-sections. A normal stress is produced in each longitudinal fibre which is not parallel to the axis of bar, and hence there is a stress component which produces an additional torque.

THE DIFFERENTIAL EQUATIONS

The governing differential equations for the coupled torsional and longitudinal vibrations of a pretwisted slender beam are

$$ES\frac{\partial^2 h}{\partial x^2}+EI_0\,\beta\,\frac{\partial^2 \alpha}{\partial x^2}-\rho S\,\frac{\partial^2 h}{\partial t^2}=0$$

$$(GJ+EI_1\,\beta^2)\,\frac{\partial^2 \alpha}{\partial x^2}+EI_0\beta\,\frac{\partial^2 h}{\partial x^2}-\rho I_0\,\frac{\partial^2 \alpha}{\partial t^2}=0 \qquad (1)$$

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where

S=the area of the cross-section of the beam

G=shear modulus

J=a constant depending upon the cross-section of the beam

β=the twist in radians per unit length of x-axis

E=the Young's modulus of elasticity

α=the angle through which each cross-section rotates

h=the displacement in the positive x-direction

ρ=the density of the fluid.

Also

$$I_0 = \int_s r^2\, ds,\ I_1 = \int_s r^4\, ds.$$

r is the distance of a longitudinal fibre from the axis of the bar.

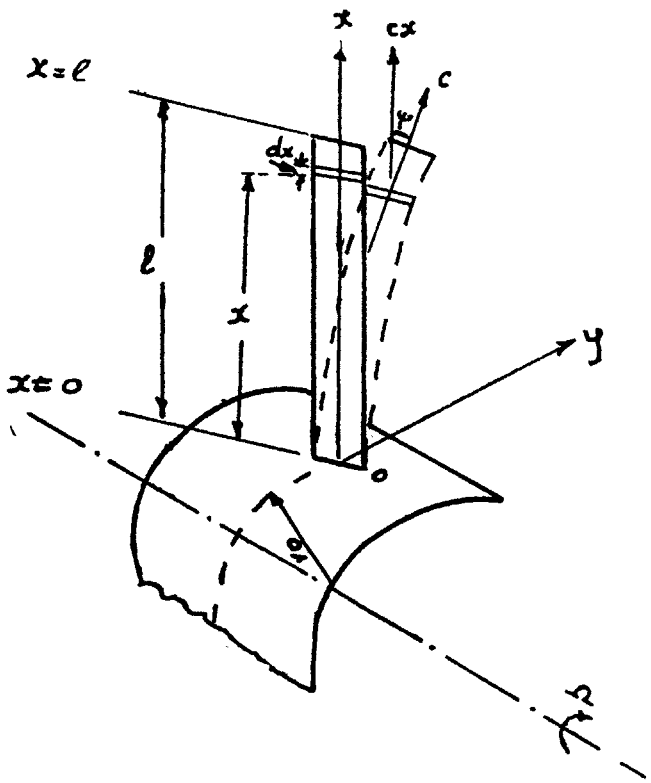

FIG. 1

When the elastic force is included, the blade will be deformed. Also in a steady flow of speed U, the blade will have some deformation due to the aerodynamic force. Then the above differential equations become

$$ES \frac{\partial^2 h}{\partial x^2} + EI_0 \beta \frac{\partial^2 \alpha}{\partial x^2} - \rho S \frac{\partial^2 h}{\partial t^2} + L = 0$$

$$(GJ + EI_1 \beta^2) \frac{\partial^2 \alpha}{\partial x^2} + EI_0 \beta \frac{\partial^2 h}{\partial x^2} - \rho I_0 \frac{\partial^2 \alpha}{\partial t^2} + N = 0 \qquad (2)$$

These L and N are given by

$$L = \frac{\rho U^2}{2} c\, C_L \text{ and } N = \frac{\rho U^2}{2} c^2 \left[C_N + \frac{x_0}{c} C_2 \right] \qquad (3)$$

where C_L and C_N are the lift and moment coefficients about the leading edge and which are expressed as given by Sharma (*in press*).

Referring to Fig. 1 the centrifugal force dc exerted by a mass element of the radial length $d\eta$ at $x = \eta$ is

$$dc = \rho S\, \Omega^2 (r_0 + \eta).$$

If the longitudinal displacement of the blade at time t is $h\,(x, t)$, the radial component dc is

$$dC_x = dc \cos \psi = dc \frac{\Delta h}{\Delta x} = dc \frac{\partial h}{\partial x} \text{ as } \Delta x \to 0.$$

Therefore,

$$C_x = \int_x^l \rho S\, \Omega^2 (r_0 + \eta)\, d\eta \frac{\partial h}{\partial x}$$

$$= \rho S\, \Omega^2 \{ r_0 (l - x) + \frac{1}{2} (l^2 - x^2) \} \frac{\partial h}{\partial x}. \qquad (4)$$

The first of eqn. (2) represents equilibrium for axial force and, therefore, it should be modified by adding C_x if the centrifugal force effect is being considered and hence then the governing differential equations become

$$ES \frac{\partial^2 h}{\partial x^2} + EI_0 \beta \frac{\partial^2 \alpha}{\partial x^2} - \rho S \frac{\partial^2 h}{\partial t^2} + L$$

$$+ \rho S\, \Omega^2 \{ r_0 (l - x) + \frac{1}{2} (l^2 - x^2) \} \frac{\partial h}{\partial x} = 0$$

$$(GJ + EI_1 \beta^2) \frac{\partial^2 \alpha}{\partial x^2} + EI_0 \beta \frac{\partial^2 h}{\partial x^2} - \rho I_0 \frac{\partial^2 \alpha}{\partial t^2} + N = 0. \qquad (5)$$

Determination of Natural Frequencies and Critical Speeds

The equations (5) are now put in terms of dimensionless variable by introducing $\xi = \frac{x}{c}$ and the substitutions

$$\alpha' = \frac{E}{\rho l^2},$$

$$m = \frac{GJ + EI_1\, \beta^2}{\rho I^0\, l^2}, \quad \nu = \Omega^2\, l, \quad \delta = \frac{EI_0\, \beta}{\rho S l^2}, \quad n = \frac{E\beta}{\rho l^2}$$

$$k_1 = \frac{\rho C}{2\rho S}\,\frac{dC_L}{d\alpha}, \quad k_2 = \left(\frac{3}{4}\, c - x_0\right)$$

$$k_3 = \frac{\rho C^2}{2\rho I_0}\,\frac{C\pi}{\delta}, \quad k_4 = \left(\frac{x_0}{C} - \frac{1}{4}\right) C$$

are used,

where,

x_0 = the distance of elastic axis after the leading edge

C = the chord length of the cross-section (air foil)

l = length of the beam.

The equations (5) become

$$\left.\begin{aligned} &\alpha' \frac{\partial^2 h}{\partial \xi^2} + \nu \left\{\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\,\xi - \tfrac{1}{2}\xi^2\right\}\frac{\partial h}{\partial \xi} - \frac{\partial^2 h}{\partial t^2} + \delta\,\frac{\partial^2 \alpha}{\partial \xi^2} \\ &\qquad + k_1 U^2 \left[\alpha + \frac{1}{U}\,\frac{\partial h}{\partial t} + \frac{k_2}{U}\,\frac{\partial \alpha}{\partial t}\right] = 0 \\ &n\,\frac{\partial^2 h}{\partial \xi^2} + m\,\frac{\partial^2 \alpha}{\partial \xi^2} - \frac{\partial^2 \alpha}{\partial t^2} - k_3 U\,\frac{\partial \alpha}{\partial t} + k_1 k_4 \left[U^2 \alpha + U\,\frac{\partial h}{\partial t} + U k^2\,\frac{\partial \alpha}{\partial t}\right] = 0. \end{aligned}\right\} \quad (6)$$

The solutions of equations (6) are of the form

$$\left.\begin{aligned} h(\xi, t) &= A\, f(\xi)\, e^{i\omega t} \\ \alpha(\xi, t) &= B\, \phi(\xi)\, e^{i\omega t} \end{aligned}\right\} \quad (7)$$

A and B are the constants which are not independent, and $f(\xi)$ and $\phi(\xi)$ are functions of ξ only.

The functions $f(\xi)$ and $\phi(\xi)$ satisfy all the boundary conditions of the beam which are

$$\left.\begin{aligned} h = \alpha &= 0 \qquad \text{at } \xi = 0 \\ \frac{\partial h}{\partial \xi} = \frac{\partial \alpha}{\partial \xi} &= 0 \qquad \text{at } \xi = 1. \end{aligned}\right\} \quad (8)$$

For an approximate determination of the fundamental frequency $f(\xi)$ is chosen as the shape function for the fundamental mode of uncoupled longitudinal vibration and $\phi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration of a uniform cantilever beam. These shape functions will satisfy the boundary condition (7) and are

$$\left.\begin{aligned} f(\xi) &= 2\xi - 3\xi^2 + \tfrac{4}{35}\xi^4 \\ \phi(\xi) &= \sin(\pi/2)\,\xi. \end{aligned}\right\} \tag{9}$$

Substituting equations (7) in (6), we obtain

$$\left.\begin{aligned} &A\left[\alpha'\frac{\alpha^2 f}{d\xi^2} + \nu\left\{\frac{r_0}{l} + \frac{1}{2} - \frac{r_0}{l}\xi - \frac{1}{2}\xi^2\right\}\frac{df}{d\xi} + f\omega^2 + k_1\, U f i\omega\right] + \\ &\qquad + B\left[\delta\frac{d^2\phi}{d\xi^2} + k_1\, U^2\phi + k_1 k_2\, U i\omega\phi\right] = 0 \\ &A\left[n\frac{d^2 f}{d\xi^2} + k_1 k_4 f U i\omega\right] + B\left[m\frac{d^2\phi}{d\xi^2} + \phi\,\omega^2 - k_3\, U i\omega\phi + \right. \\ &\qquad \left. + k_1 k_4\, U^2\phi + k_1 k_2 k_4\, U i\omega\phi\right] = 0. \end{aligned}\right\} \tag{10}$$

Equations (10) can be solved for ω^2 but the result is a function of ξ, since f and ϕ are not the exact shape functions. This difficulty can be overcome by multiplying the first and second of equations (10) by f and ϕ respectively and integrating the result with respect to ξ from 0 to 1, wə obtain the following equations.

$$\left.\begin{aligned} &A\left[-a_1 + a_2 + a_3\omega^2 + a_4\, U i\omega\right] + B\left[-a_5 + a_6 U^2 + a_7\, U i\omega\right] = 0 \\ &A\left[-a_8 + U i\omega a_9\right] + B\left[-a_{10} + \omega^2 a_{11} - U i\omega_{12} + U^2 a_{13} + U i\omega_{14}\right] = 0 \end{aligned}\right\} \tag{11}$$

where

$$a_1 = -\alpha'\int_0^1 \frac{d^2 f}{d\xi^2}\, f d\xi = \alpha'\int_0^1 \left(\frac{df}{d\xi}\right)^2 d\xi$$

$$a_2 = \nu\int_0^1 \left(\frac{r_0}{l} + \frac{1}{2} - \frac{r_0}{l}\xi - \frac{1}{2}\xi^2\right)\frac{df}{d\xi}\, f d\xi$$

$$a_3 = \int_0^1 f^2\, d\xi$$

$$a_4 = k_1\int_0^1 f^2 d\xi$$

$$a_5 = -\delta\int_0^1 \frac{d^2\phi}{d\xi^2} f d\xi = \delta\int_0^1 \frac{d\phi}{d\xi}\frac{df}{d\xi}\, d\xi\,.$$

$$a_6 = k_1\int_0^1 f\phi d\xi$$

$$a_7 = k_1 k_2 \int_0^1 f\phi\, d\xi$$

$$a_8 = -n\int_0^1 \frac{d^2f}{d\xi^2}\,\phi\, d\xi = n\int_0^1 \frac{df}{d\xi}\frac{d\phi}{d\xi}\, d\xi.$$

$$a_9 = k_1 k_1 \int_0^1 f\phi d\xi.$$

$$a_{10} = -m\int_0^1 \frac{d^2\phi}{d\xi^2}\,\phi d\xi = m\int_0^1 \left(\frac{d\phi}{d\xi}\right)^2 d\xi.$$

$$a_{11} = \int_0^1 \phi^2\, d\xi$$

$$a_{12} = k_3\int_0^1 \phi^2\, d\xi$$

$$a_{13} = k_1 k_4 \int_0^1 \phi^2\, d\xi$$

$$a_{14} = k_1 k_2 k_4 \int_0^1 \phi^2\, d\xi.$$

For a non-trivial solution, A and B must not both vanish consequently the determinant of the coefficient of equation must be zero. This determinant being complex, both the real and imaginary parts must vanish. On setting the determinant equal to zero

$$\begin{vmatrix} -a_1+a_2+\omega^2 a_{13}+iU\omega_4 & -a_5+U^2a_6+Ui\omega a_7 \\ -a_8+Ui\omega a_9 & \begin{array}{l} -a_{10}+\omega^2 a_{11}-i\omega U a_{12} \\ \quad +U^2 a_{13}+Ui\omega a_{14} \end{array} \end{vmatrix} = 0$$

i. e

$$\left.\begin{array}{l} A_1\omega^4-(C_1+C_2U^2)\,\omega^2+(E_1+E_2U^2) = 0 \\ -B_1\omega^2+B_2+B_3U^2 \qquad\qquad\qquad\; = 0 \end{array}\right\} \tag{12}$$

where

$$A_1 = a_3 a_{11}$$

$$C_1 = a_3 a_{10} + (a_1 - a_2) a_{11}$$

$$C_2 = -a_3 a_{13} - a_4 a_{12} + a_4 a_{14} - a_7 a_9$$

$$E_1 = a_{10} (a_1 - a_2) - a_5 a_8$$

$$E_2 = -a_{13} (a_1 - a_2) + a_6 a_8$$

$$B_1 = -a_4 a_{11} + a_3 a_{12} - a_3 a_{14}$$

$$B_2 = (a_1 - a_2)(a_{12} - a_{14}) + a_7 a_8 + a_5 a_9 - a_4 a_{10}$$

$$B_3 = a_4 a_{13} - a_6 a_9 .$$

The second equation of (12) gives,

$$\omega^2 = \frac{B_2 + B_3 U^2}{B_1} . \tag{13}$$

Substituting this expression into the first equation of (12) we obtain

$$PU^4 - QU^2 + R = 0 \tag{14}$$

where

$$P = B_3 (B_1 C_2 - A_1 B_3)$$

$$Q = 2B_2 B_3 A_1 - B_1 B_2 C_2 - C_1 B_1 B_3 + E_2 B_1^2$$

$$R = B_1 (B_2 C_1 - E_1 B_1) - A_1 B_2^2 .$$

From equation (13), we obtain the value of the critical speed as given below

$$U^2 = \frac{Q \pm \sqrt{Q^9 - 4PR}}{2P} \tag{15}$$

The right hand side of (15) is positive. Corresponding to the solutions of U^2 from eqn. (15), there are two values of ω^2 from eqn. (13). Usually the smaller U^2 is associated with the higher value of ω^2 for in equation (13), the coefficients B_1 and B_2 are always positive whereas B_3 is negative if the elastic axis lies behind the 1/4-chord point, as is usually so.

Numerical Example

A numerical example for the coupled torsional and longitudinal vibrations of a slender pretwisted and rotatory beam under aerodynamic coupling is now presented. The frequencies and speeds are computed from the eqns. (13) and (15) respectively. The cross-section of the beam is taken as a rectangle having width $2b$ and depth $2h'$. The formula $J = 4k_1 sh'^2$ is used to calculate J, where $k_1 = 0{\cdot}312$ for $h'/10 = 1/10$, and $1/3$ for $h'/6 = 1/15$ and $1/20$. (Timoshenko and Goodier 1951).

The other physical constants of the beam are taken as follows :

$$E = 30 \times 10^6 \ \text{lb/in}^2$$

$$G = 11{\cdot}25 \times 10^6 \ \text{lbs in}^2$$

$$r_0 = 32 \text{ in}$$

$$\Omega = 314 \ \text{sec}^{-1}$$

$$l = 8 \text{ in}$$

$$b = 1 \text{ in}$$

$$\rho = {\cdot}00082 \ \text{lb/in}^3$$

$$C = {\cdot}24885$$

$$\frac{dC_L}{d\alpha} = 6 \text{ radians} < 2\pi.$$

With these values, the various parameters used in the frequency and speed equations are as follows

$$a_1 = 30.478768 \times 10^7$$

$$a_2 = 0{\cdot}021722 \times 10^7$$

$$a_3 = 0.120635$$

$$a_4 = 0.034363$$

$$a_5 = 1.102347 \times 10^7$$

$$a_6 = 0.066734$$

$$a_7 = 0.051644$$

$$a_8 = 0.330704 \times 10^8$$

$$a_9 = 0.055455$$

$$a_{10} = 1.414446 \times 10^7$$

$$a_{11} = 0.5$$

$$a_{12} = 0.335720$$

$$a_{13} = 0.118355$$

$$a_{14} = 0.091592.$$

With these values, we obtain the critical speeds

$$U_1^2 = -1.025791 \times 10^{10} \text{ and } U_2^2 = -1.642914 \times 10^{11}.$$

Now substituting these values in (13), we obtain

$$\omega_1^2 = 5.906006 \times 10^9$$

and

$$\omega_2^2 = 1.316140 \times 10^9$$

which are the upper bounds of the frequencies corresponding to the first two modes of coupled torsional and longitudinal vibrations of a pretwisted rotating slender beam under aero-dynamic coupling.

References

Timoshenko, S., and Goodier, J. N. (1951). Theory of Elasticity. McGraw-Hill Book Co, Inc., New York. p. 277.

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