GREEN'S FUNCTION OF ELASTIC PLATE IN THE SHAPE OF EPITROCHOID OF ORDER-3 AND CLAMPED ALONG THE BOUNDARY

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An expression, in terms of an infinite convergent series is found for the transverse displacement due to an isolated load in a thin elastic plate in the shape of epitrochoid of order three, and clamped along the boundary. The region can be mapped conformally onto a unit circle. The method of evaluation of the coefficients occurring in the series is not direct since the number of unknown coefficients is greater than the number of equations. The coefficients can be determined in terms of three of unknown coefficients and then these unknown coefficients can be determined by applying a condition of convergence (Singh 1960). The numerical evalution has been done for a few simple cases.

1. INTRODUCTION

The plate is bounded externally by, and is clamped along, the epitrochoid (Fig. 1) whose parametric equation is given by (Mushkhelisvilli 1963).

$$\left.\begin{aligned} x &= R\,(\rho \cos\phi + m\rho^4 \cos 4\phi) \\ y &= R\,(\rho \sin\phi + m\rho^4 \sin 4\phi) \end{aligned}\right\} \tag{1}$$

giving

$$\frac{r^2}{R^2} = \rho^2 + m^2\rho^8 + 2m\rho^5 \cos 3\phi \tag{1a}$$

The interior of the epitrochoid can be represented conformally on the circle of radius $a \leq 1$ in the ζ plane (Fig. 2) by the transformation (Mushkhelisvilli 1963)

$$z = x + iy = R\,(\zeta + m\zeta^4) \tag{2}$$

where $R > 0,\ 0 \leq m \leq \frac{1}{4}$

and $\zeta = \xi + i\eta = \rho e^{i\phi}$ (3)

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A transverse force F is supposed to be applied at the point $Q(z_0)$, and w denotes the small transverse displacement of the plate at the point $P(z)$; w is then a biharmonic function of the variables x, y and (apart from a factor) sum of

$$|z-z_0|^2 \log|z-z_0|$$

and of a function with no singularity at Q (Love 1952).
Now

$$z-z_0 = R[(\zeta-\zeta_0)\{1+m(\zeta+\zeta_0)(\zeta^2+\zeta_0^2)\}]$$

and there is no singularity of

$$\log R\{1+m(\zeta+\zeta_0)(\zeta^2+\zeta_0^2)\} \quad \text{near } \zeta=\zeta_0.$$

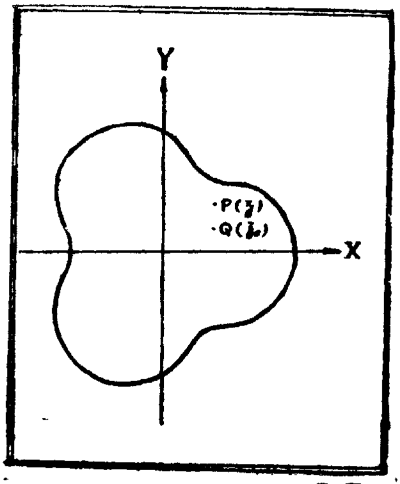

FIG. 1

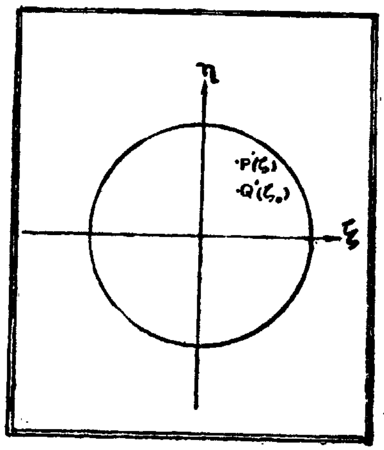

FIG. 2

Hence we can take the Green's function to be the sum of

$$|z-z_0|^2 \log|\zeta-\zeta_0|$$

and a biharmonic function with no singularity at Q.

The boundary conditions require that the displacement w and its normal derivative $\frac{\partial w}{\partial n}$ must vanish at the clamped boundary.

It is convenient to divide the problem of finding the displacement w due to a transverse force at an arbitrary point $Q(x_0, y_0)$ of the plate into two parts :

(*i*) to determine the displacement w_1 caused by two forces each equal to F at the point $Q(x_0, y_0)$ and $Q_1(x_0, -y_0)$

(*ii*) to determine the displacement w_2 caused by a force F at $Q\,(x_0, y_0)$ and an opposite force $(-F)$ at the point $Q_1\,(x_0, -y_0)$.

Here w_1 and w_2 satisfy the same boundary conditions as w. The Green's function g and the actual displacement w at $P\,(x, y)$ for a transverse force F at $Q\,(x_0, y_0)$ are given by

$$g = \tfrac{1}{2}\,(g_1+g_2)$$

$$w = \tfrac{1}{2}\,(w_1+w_2)$$

where

$$w_r = \frac{F}{8\pi D}\, g_r, \quad r = 1, 2 \tag{4}$$

It is clear from symmetry that w_1 and w_2 are respectively even and odd functions of ϕ.

2. Determination of g_1

To determine w_1 in § 1 (*i*) we start with

$$g'_1 = |\,z-z_0\,|^2 \log|\,\zeta-\zeta_0\,| + |\,z-\bar{z}_0\,|^2 \log|\,\zeta-\bar{\zeta}_0\,|, \tag{5}$$

where $z, z_0, \bar{z}_0$ are the complex coordinates of P, Q, Q_1 respectively in the z-plane. Let $\zeta_0 = c \exp(i\lambda)$ be the image of the point z_0.

We have for $1 \geqslant \rho \geqslant c \geqslant 0$,

$$|\,z-z_0\,|^2 \log|\,\zeta-\zeta_0\,| = \sum_{n=0}^{\infty} P_n \cos n\phi + \sum_{n=1}^{\infty} Q_n \sin n\phi \tag{6}$$

where it can be shown that if $u_n = \dfrac{1}{n}\left(\dfrac{c}{\rho}\right)^n$,

$$\begin{aligned} P_0 = R^2 \big[&\{(\rho^2+c^2+m^2\rho^8+m^2c^8)-\log\rho+\rho c u_1+m^2\rho^4c^4u_4\} \\ &+\{2mc^5\log\rho+m\rho c^4u_1-m\rho^5u_3+mc\rho^4u_4\}\cos 3\lambda\big] \end{aligned} \tag{7a}$$

$$\begin{aligned} P_1 = R^2 \big[&\{\rho c u_2+m^2\rho^4c^4\,(u_3+u_5)-2\rho c\log\rho \\ &-(\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_1\}\cos\lambda+\{m\rho\,(c^4-\rho^4)\,u_2 \\ &+mc\rho^4u_3-mc^5u_1\}\cos 2\lambda+\{mc\rho^4u_5-mc^5u_1 \\ &-2m\rho c^4\log\rho-m\rho^5\,u_4\}\cos 4\lambda\big] \end{aligned} \tag{7b}$$

$$\begin{aligned} P_2 = R^2 \big[&(m\rho c^4u_3-m\rho^5u_1+mc\rho^4u_2-mc^5u_2)\cos\lambda \\ &+\{\rho c\,(u_1+u_3)+m^2\rho^4c^4\,(u_2+u_6)-(\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_2\}\cos 2\lambda \\ &+\{m\rho c^4u_1-m\rho^5u_5+mc\rho^4u_6-mc^5u_2\}\cos 5\lambda\big] \end{aligned} \tag{7c}$$

$$\begin{aligned} P_3 = R^2 \big[&\{mc\rho^4u_1+m\rho c^4u_4-mc^5u_3+2m\rho^5\log\rho\}+\{\rho c\,(u_2+u_4) \\ &+m^2\rho^4c^4\,(u_1+u_7)-(\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_3\}\cos 3\lambda+\{m\rho c^4u_2 \\ &+mc\rho^4u_7-m\rho^5u_6-mc^5u_3\}\cos 6\lambda\big] \end{aligned} \tag{7d}$$

$$\begin{aligned} P_4 = R^2 \big[&\{m\rho c^4u_5-m\rho^5u_1-mc^5u_4-2mc\rho^4\log\rho\}\cos\lambda+\{\rho c\,(u_3+u_5) \\ &-(\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_4-2m^2\rho^4c^4\log\rho+m^2\rho^4c^4u_8\}\cos 4\lambda \\ &+\{m\rho c^4u_3-m\rho^5u_7+mc\rho^4u_8-mc^5u_4\}\cos 7\lambda\big] \end{aligned} \tag{7e}$$

and for $n \geqslant 5$

$$P_n = R^2 [\{m\rho c^4 u_{n+1} + mc\rho^4 u_{n-4} - m\rho^5 u_{n-3} - mc^5 u_n\} \cos \overline{n-3}\,\lambda + \{\rho c\,(u_{n-1}+u_{n+1}) + m^2\rho^4c^4\,(u_{n-4}+u_{n+4}) - (\rho^2+c^2+m^2\rho^8+m^2c^8)u_n\} \times \cos n\lambda + \{m\rho c^4 u_{n-1} - mc^5 u_n + mc\rho^4 u_{n+4} - m\rho^5 u_{n+3}\} \cos \overline{n+3}\,\lambda \tag{7f}$$

Also

$$Q_1 = R^2 [\{\rho c u_2 + m^2\rho^4c^4\,(u_3+u_5) - 2\rho c \log \rho - (\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_1\} \sin \lambda - \{m\rho\,(c^4-\rho^4)\,u_2 + mc\rho^4 u_3 - mc^5 u_1\} \sin 2\lambda + \{mc\rho^4 u_5 - mc^5 u_1 - 2m\rho c^4 \log \rho - m\rho^5 u_4\} \sin 4\lambda] \tag{8a}$$

$$Q_2 = R^2 [-\{m\rho c^4 u_3 - m\rho^5 u_1 + mc\rho^4 u_2 - mc^5 u_2\} \sin \lambda + \{\rho c\,(u_1+u_3) + m^2\rho^4c^4\,(u_2+u_6) - (\rho^2+c^2+m^2\rho^8+m^2\,c^8)\,u_2\} \sin 2\lambda + \{m\rho c^4 u_1 - m\rho^5 u_5 + mc\rho^4 u_6 - mc^5 u_2\} \sin 5\lambda] \tag{8b}$$

$$Q_3 = R^2 [\{\rho c\,(u_2+u_4) + m^2\rho^4c^4\,(u_1+u_7) - (\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_3\} \sin 3\lambda + \{m\rho c^4 u_2 + mc\rho^4 u_7 - m\rho^5 u_6 - mc^5 u_3\} \sin 6\lambda] \tag{8c}$$

$$Q_4 = R^2 [\{m\rho c^4 u_5 - m\rho^5 u_1 - mc^5 u_4 - 2mc\rho^4 \log \rho\} \sin \lambda + \{\rho c\,(u_3+u_5) - (\rho^2+c^2+m^2\rho^8+m^2c^8)\,u_4 - 2m^2\rho^4c^4 \log \rho + m^2\rho^4c^4 u_8\} \sin 4\lambda + \{m\rho c^4 u_3 - m\rho^5 u_7 + mc\rho^4 u_8 - mc^5 u_4\} \sin 7\lambda] \tag{8d}$$

and for $n \geqslant 5$

$$Q_n = R^2 [\{m\rho c^4\,u_{n+1} + mc\rho^4 u_{n-4} - m\rho^5 u_{n-3} - mc^5 u_n\} \sin \overline{n-3}\,\lambda + + \{\rho c\,(u_{n-1}+u_{n+1}) + m^2\rho^4c^4\,(u_{n-4}+u_{n+4}+4) - (\rho^2+c^2m^2\rho^8+m^2c^8)\,u_n\} \sin n\lambda + + \{m\rho c^4 u_{n-1} - mc^5 u_n + mc\rho^4 u_{n+4} - m\rho^5 u_{n+3}\} \sin \overline{n+3}\,\lambda] \tag{8e}$$

Changing the sign of λ in (6), we have

$$|\,z-\bar{z}_0\,|^2 \log |\,\zeta-\bar{\zeta}_0\,| = \sum_{n=0}^{\infty} P_n \cos n\phi - \sum_{n=1}^{\infty} Q_n \sin n\phi \tag{9}$$

and therefore,

$$g_1' = 2 \sum_{n=0}^{\infty} P_n \cos n\phi \tag{10}$$

the P's being given by eqns. (7*a*)-(7*f*).

It follows that

$$(g_1')_{\rho=a} = \sum_{n=0}^{\infty} A_n \cos n\phi$$

and

$$\left(\frac{\partial g_1'}{\partial \rho}\right)_{\rho=a} = \sum_{n=0}^{\infty} A_n' \cos n\phi \tag{11}$$

The corresponding solution of Brans-Dicke theory is given by

$$\left.\begin{aligned}
\Phi &= (1-mx)^{Am/p}, \quad p = \left(2\omega + \frac{6\pi}{k}\right)^{1/2} \\
\bar{F}_{14} &= \left(\frac{8\pi G_0}{k}\right)^{1/2}\left(\frac{Mm}{1-mx}\right)\operatorname{sech}^2\{M\log(1-mx)\} \\
\bar{g}_{11} &= \bar{g}_{22} = \bar{g}_{33} = -(1-mx)^{2-\frac{AM}{p}}\cosh^2\{M\log(1-mx)\} \\
\bar{g}_{44} &= (1-mx)^{-\frac{AM}{p}}\operatorname{sech}^2\{M\log(1-mx)\}.
\end{aligned}\right\} \qquad (17)$$

(ii) *A Plane Symmetric Static Solution*

The plane symmetric static solution of Einstein's empty space field equations obtained by Taub (1951) is given by the metric,

$$\begin{aligned}
ds^2 &= (k_1x+k_2)^{-1/2}(dt^2-dx^2) - (k_1x+k_2)(dy^2+dz^2) \\
&= (k_1x+k_2)^{-1/2}\,dt - (k_1x+k_2)^{1/2}\{(k_1x+k_2)^{-1}\,dx^2 \\
&+ (k_1x+k_2)^{1/2}(dy^2+dz^2)\}
\end{aligned} \qquad \ldots \quad (18)$$

where k_1 and k_2 are constants. This solution corresponds to the gravitational field of an infinite plane parallel to the (y, z)-plane.

Here

$$v = -(1/4)\log(k_1 x + k_2) \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

and

$$h_{ij}\,dx^i\,dx^j = (k_1x+k_2)^{-1}\,dx^2 + (k_1x+k_2)^{1/2}(dy^2+dz^2) \quad \ldots \quad \ldots \quad (20)$$

Thus in this case the scalar field ψ is given by

$$\psi = AM\log(k_1x+k_2) \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

where $M = -\dfrac{1}{4}\left(1+\dfrac{kA^2}{8\pi}\right)^{-1/2}$. Also

$$u = M\log(k_1x+k_2).$$

Again from (5), the only non-zero component of the electromagnetic field tensor F_{ij} is

$$F_{14} = -\left(\frac{8\pi}{k}\right)^{1/2}\left(\frac{(M\,k_1)}{k_1x+k_2}\right)\operatorname{sech}^2\{M\log(k_1x+k_2)\}. \ldots \quad \ldots \quad (22)$$

The equations to determine the coefficients J_n, L_n $(n \geqslant 0)$ are obtained by substituting (11), (16) and (17) in (13) and in equating the coefficients of $\cos n\phi$, $(n \geqslant 0)$ we have

$$\begin{aligned} a^2 J_1 + m a^8 J_4 + L_0 &= A_0 \\ 2a^2 J_1 + 8m a^8 J_4 &= a A'_0 \end{aligned} \tag{19}$$

$$\begin{aligned} a^3 J_2 + ma^7 J_3 + ma^9 J_5 + a L_1 &= A_1 \\ 3a^3 J_2 + 7m a^7 J_3 + 9m a^9 J_5 + a L_1 &= a A'_1 \end{aligned} \tag{20}$$

$$\begin{aligned} a^4 J_3 + m a^6 J_2 + m a^{10} J_6 + a^2 L_2 &= A_2 \\ 4a^4 J_3 + 6m a^6 J_2 + 10m a^{10} J_6 + 2a^2 L_2 &= a A'_2 \end{aligned} \tag{21}$$

$$\begin{aligned} a^5 J_4 + m a^5 J_1 + m a^{11} J_7 + a^3 L_3 &= A_3 \\ 5a^5 J_4 + 5m a^5 J_1 + 11m a^{11} J_7 + 3a^3 L_3 &= a A'_3 \end{aligned} \tag{22}$$

for $n \geqslant 4$,

$$\begin{aligned} a^{n+2} J_{n+1} + m a^{n+8} J_{n+4} + a^n L_n &= A_n \\ (n+2)\, a^{n+2} J_{n+1} + m\,(n+8)\, a^{n+8} J_{n+4} + n a_n L_n &= a A'_n. \end{aligned} \tag{23}$$

The solution of these equations is not direct, for the number of unknown coefficients is greater than the number of equations. From these sets of equations it is seen that $J_4, J_7, J_{10}, J_{13}, \ldots\ldots, L_0, L_3, L_6. L_9, \ldots\ldots$ are all determined in terms of A_{3n}, A'_{3n} $(n=0, 1, 2\ldots\ldots)$ together with J_1.

Similarly $J_5, J_8, J_{11}, J_{14}, \ldots\ldots, L_1, L_4, L_7, L_{10} \ldots \ldots$ are all determined in terms of A_{3n+1}, A'_{3n+1} $(n=0, 1, 2\cdots\cdots)$ together with an expression (say U_2) containing J_2 and J_3.

Also $J_6, J_9, J_{12} \ldots\ldots, L_2, L_5, L_8, \ldots\ldots$ are all determined in terms of A_{3n+2}, A'_{3n+2} $(n=0, 1, 2\ldots\ldots)$ together with another expression (say U_3) containing J_2 and J_3.

The correct value of J_1, U_2, U_3 are determined from a condition of convergence (Singh (1960).

Finally solving U_2 and U_3 simultaneously the correct value of J_2 and J_3 are determined. Thus knowing J_1, J_2 and J_3 all the coefficients J_n, L_n $(n \geq 0)$ are uniquely determined by the sets of equations (19)-(23).

Equations (19) give

$$8\,ma^8 J_4 = a A_0' - 2a^2 J_1$$
$$8 L_0 = 8 A_0 - a A_0' - 6 a^2 J_1. \tag{24}$$

Equations (20) give

$$8\,ma^9 J_5 = a A_1' - A_1 - 2a^3 (J_2 + 3ma^4 J_3)$$
$$8\,aL_1 = 9 A_1 - a A_1' - 2a^3 (3J_2 + ma^4 J_3). \tag{25}$$

Equations (21) give

$$8\,m a^{10} J_6 = a A_2' - 2A_2 - 2a^4 (J_3 + 2ma^2 J_2)$$
$$8 a^2 L_2 = 10 A_2 - a A_2' - 2a^4 (3 J_3 + 2m a^2 J_2) \tag{26}$$

Equations (22) give

$$8m a^{11} J_7 = a A_3' - 3A_3 - 2a^5 (J_4 + mJ_1)$$
$$8a^3 L^3 = 11 A_3 - a A_3' - 6a^5 (J_4 + mJ_1). \tag{27}$$

Equations (23) give, $n \geq 4$,

$$8m a^{n+8} J_{n+4} = aA_n' - n A_n - 2a^{n+2} J_{n+1}$$
$$8 a^n L_m = (n+8) A_n - aA_n' - 6 a^{n+2} J_{n+1}. \tag{28}$$

Thus the first equation in (24), (27) and (28) gives J_4. J_7, J_{10}, ...J_{4n+1}......in terms of A_{3n}, A_{3n}', $(n=0, 1, 2, \ldots..)$ and J_1. Similarly the first equation in (25) and (28) gives J_5, J_8,...... J_{3n-1}... in terms of A_{3n+1}, A_{3n+1}' $(n=0,1.2...)$ and U_2 $(=J_2 + 3ma^4 J_3)$ and so also the first equation in (26) and (28) gives J_6, J_9, J_{12}, ... J_{3n} ..in terms of A_{3n+2}, A_{3n+2} $(n=0, 1, 2\cdots\cdots)$ *and* U_3 $(=J_3 + 2m a^2 J_2)$.

Now putting

$$\overset{*}{J}^n = a^n J_n$$
$$\overset{*}{L}_n = a^n L_n.$$

We have

$$8m a^4 \overset{*}{J}_{n+4} = a A_n' - 2a \overset{*}{J}_{n+1} \tag{29a}$$
$$8\overset{*}{L}_n = (n+8) A_n - aA_n' - 6a \overset{*}{J}_{n+1}. \tag{29b}$$

For the convergence of the series (16) it is necessary that

$$\overset{*}{J}_{n+4} \to 0 \text{ and } \overset{*}{L}_n \to 0 \text{ as } n \to \infty.$$

The terms in (29 *a*) and (29 *b*) which depend on A_n, A'_n. tend to zero because of the convergence of the original series (11). If these terms are left out for large *n*,

$$\overset{*}{J}_{n+4} \doteq -\frac{J^*_{n+1}}{4m\,a^3}$$

$$\overset{*}{L}_n \doteq -\frac{3a\,J^*_{n+1}}{4}\,. \tag{30}$$

Since $a < 1$, in solving the first equation of (24), (27) and (28) for J^*_{3n+1} ($n=1, 2\ldots\ldots$) it is coefficient J^*_{3n+1} that may not tend to zero unless J_1 has correct value. Similarly, in solving the first equation of (26) and (28) for J^*_{3n} ($n=2, 3\ldots$) and the first equation of (25) and (28) for J^*_{3n-1} ($n=2, 3,\cdots$) the coefficients J^*_{3n} and J^*_{3n-1} may not tend to zero unless U_2 and U_3 have correct values.

The approximate value of J_1 is determined by taking $J^*_{3n+1}=0$. and those of U_2 and U_3 are determined by taking $J^*_{3n}=0$ and $J^*_{3n-1}=0$ simultaneously. From the correct values of U_2 and U_3, those of J_2 and J_3 are calculated.

Calculating numerically for $a = \frac{3}{5}$, $c = \frac{1}{2}$, $m = \frac{1}{5}$, $\lambda=0$ and $R=1$, it is seen that

$$\begin{aligned}
J_4 &= 0 \text{ gives } J_1 = -0.685416\\
J_7 &= 0 \text{ gives } J_1 = -0.655788\\
J_{10} &= 0 \text{ gives } J_1 = -0.691380\\
J_{13} &= 0 \text{ gives } J_1 = -0.690849\\
J_{16} &= 0 \text{ gives } J_1 = -0.690904
\end{aligned}$$

From the same set of values *a*, *c*, *m*, λ and *R* it is also seen that

$$\begin{aligned}
J_5 &= 0 \text{ gives } U_2 = -1.042655\\
J_8 &= 0 \text{ gives } U_2 = -1.050657\\
J_{11} &= 0 \text{ gives } U_2 = -1.043796\\
J_{12} &= 0 \text{ gives } U_2 = -1.044544\\
J_{17} &= 0 \text{ gives } U_2 = -1.044468
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_6 &= 0 \text{ gives } U_3 = 0.000270\\
J_9 &= 0 \text{ gives } U_3 = -0.085439\\
J_{12} &= 0 \text{ gives } U_3 = -0.075544\\
J_{15} &= 0 \text{ gives } U_3 = -0.076597\\
J_{18} &= 0 \text{ gives } U_3 = -0.076489
\end{aligned}$$

Solving for J_2 and J_3 from $J_{17} = 0$ and $J_{18} = 0$ it is seen that

$$J_2 = -1.05028$$

$$\text{and} \quad J_3 = 0.07473$$

Hence we can take approximate values of J_1, J_2 and J_3 as 0.69090, -1.05028 and 0.07473 respectively.

3. Determination of g_2

To find the function w_2 mentioned in § 1 (*ii*), we start with

$$g'_2 = |z-z_0|^2 \log|\zeta-\zeta_0| - |z-\bar{z}_0|^2 \log|\zeta-\bar{\zeta}_0|$$

$$= 2\sum_1^{\infty} Q_n \sin n\phi \tag{31}$$

Where Q's are given by the equations (8*a*)—(8*e*)

Hence

$$(g'_2)_{\rho=a} = \sum_1^{\infty} B'_n \sin n\phi$$

$$\left(\frac{\partial g'_2}{\partial \rho}\right)_{\rho=a} = \sum_1^{\infty} B_n \sin n\phi \tag{32}$$

where

$$B_n = 2\,(Q_n)_{\rho=a}$$

and

$$B_n' = 2\left(\frac{\partial Q_n}{\partial \rho}\right)_{\rho=a} . \tag{33}$$

Proceeding exactly as in § 2, we can assume

$$g_2 = g'_2 - g''_2 \cdots \tag{34}$$

and put

$$g_2'' = \frac{1}{R}\,(xu_1 + y\,v_1) + U_1$$

where

$$u_1 = \sum_1^{\infty} k_n\,\rho^n \sin n\phi$$

$$v_1 = \sum_1^{\infty} k_n\,\rho^n \cos n\phi \tag{35}$$

$$U_1 = \sum_1^{\infty} M_n\,\rho^n \sin n\phi$$

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where we have neglected the constant term in v_1 as before. We have then

$$g''_2 = (k_2 \rho^3 - m k_3 \rho^7 + m \rho^9 k_5 + M_1 \rho) \sin \phi + (k_3 \rho^4 - m \rho^6 k_2 + + m \rho^{10} k_6 + M_2 \rho^2) \sin 2\phi + (k \rho^5 + m \rho^{11} k_7 + M_3 \rho^3) 3 \phi + + \sum_{n\geqslant 4} (k_{n+1} \rho^{n+2} + m\rho^{n+8} k_{n+4} + M_n \rho^n) \sin n \phi \tag{36}$$

$$\frac{\rho \, \partial g''_2}{\partial \rho} = (3\rho^3 k_2 - 7m k_3 \rho^7 + 9m \rho^9 k_5 + M_1 \rho) \sin \phi + (4\rho^4 k_3 - - 6m\rho^6 k_2 + 10 m \rho^{10} k_6 + 2\rho^2 M_2) \sin 2\phi + (5k \rho^5 + 11 m \rho^{11} k_7 + + 3 \rho^3 M_3) \sin 3\phi + \sum_{n\geqslant 4} \{(n+2) \rho^{n+2} k_{n+1} + m (n+8) \rho^{n+8} k_{n+4} + + n\rho^n M_n\} \sin n \phi \tag{37}$$

where $k = k_4 - m k_1$.

Hence $g_2 = \dfrac{\partial g_2}{\partial \rho} = 0$ on $\rho = a$ require

$$\left.\begin{aligned} a^3 k_2 - m a^7 k_3 + m a^9 k_5 + a M_1 &= B_1 \\ 3a^3 k_2 - 7m a^7 k_3 + 9m a^9 k_5 + a M_1 &= a B'_1 \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} a^4 k_3 - a^6 m k_2 + a^{10} m k_6 + a^2 M_2 &= B_2 \\ 4a^4 k_3 - 6m a^6 k_2 + 10m a^{10} k_6 + 2 a^2 M_2 &= a B'_2 \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} a^5 k + a^{11} m k_7 + a^3 M_3 &= B_3 \\ 5a^5 k + 11m a^{11} k_7 + 3a^3 M_3 &= a B'_3 \end{aligned}\right\} \tag{40}$$

For $n \geq 4$,

$$\left.\begin{aligned} a^{n+2} k_{u+1} + m a^{n+8} k_{n+4} + a^n M_n &= B_n \\ (n+2) a^{n+2} k_{n+1} + (n+8) m a^{n+8} k_{n+4} + n a^n M_n &= a B'_n \end{aligned}\right\} \tag{41}$$

It is seen that (41) is the same as (23)., if in the latter B's are replaced by A's.

The solution of these equations is not direct as before. They are solved exactly in the same way as in the previous section. Solving (38),

$$\left.\begin{aligned} 8 m a^9 k_5 &= a B'_1 - B_1 - 2a^3 (k_2 - 3w a^4 k_3) \\ 8a M_1 &= 9B_1 - b B'_1 - 2a^3 (3k_2 - m a^4 k_3) \end{aligned}\right\} \tag{42}$$

Solving (39),

$$\left.\begin{aligned} 8m a^{10} k_6 &= aB'_2 - 2B_2 - 2a^4 (k_3 - 2m a^2 k_2) \\ 8a^2 M_2 &= 10B_2 - bB'_2 - 2a^4 (3k_3 - 2m a^2 k_2.). \end{aligned}\right\} \tag{43}$$

Solving (40),

$$8ma^{11}k_7 = aB'_3 - 3B_3 - 2a^5k$$
$$8a^8M_3 = 11B_3 - aB'_3 - 6a^5k. \tag{44}$$

As in the previous section the correct value of K, V_2 $(=K_2 - 3ma^4 K_3)$, V_3 $(=K_3 - 2ma^2 K_2)$ are determined from the condition of convergence ; and then the correct values of K, K_3 and K_4 are determined. The values of K are calculated for $a = \frac{3}{5}$, $c = \frac{1}{2}$, $m = \frac{1}{5}$, $\lambda = \frac{\pi}{2}$, $R = 1$

$K_7 = 0$ gives $K = 0.14838$
$K_{10} = 0$ gives $K = 1.74459$
$K_{13} = 0$ gives $K = 1.71908$
$K_{16} = 0$ gives $K = 1.72206$
$K_{19} = 0$ gives $K = 1.72176.$

From the same set of values for a, c, m, λ and R, it is also seen that

$K_5 = 0$ gives $V_2 = -1.81821$
$K_8 = 0$ gives $V_2 = -1.82622$
$K_{11} = 0$ gives $V_2 = -1.81935$
$K_{14} = 0$ gives $V_2 = -1.820101$
$K_{17} = 0$ gives $V_2 = -1.82003.$

Also for the same set of values for R, a, c, m, and λ

$K_6 = 0$ gives $V_3 = -0.93249$
$K_9 = 0$ gives $V_3 = -1.01793$
$K_{12} = 0$ gives $V_3 = -1.00803$
$K_{15} = 0$ gives $V_3 = -1.00909$
$K_{18} = 0$ gives $V_3 = -1.00897.$

Solving for K_2 and K_3 from $K_{17} = 0$ and $K_{18} = 0$, we have

$$K_2 = -1.72229 \, ; \qquad K_3 = 1.25694.$$

Hence we can take the approximate values of K, K_2 and K_3 as 1.72176, -1.72229 and 1.25694 respectively.

All the unknown coefficients in g_2, as given by (36), being thus determined the function w_2 is known.

Hence the actual displacement w is given by

$$w = \frac{F}{8\pi D}\,\frac{g_1+g_2}{2}$$

$$= \frac{F}{8\pi D}\{\,|\,z-z_0\,|^2 \log|\,\zeta-\zeta_0\,| - \tfrac{1}{2}(g''_1+g''_2)\}. \tag{45}$$

4. Numerical Evaluation of Displacement

In the actual numerical evaluation of displacements, we can consider only a finite number of terms of the series u, v, U, u_1, v_1, U_1 and the conditions of the problem can be only approximately satisfied. It is, therefore, necessary to show that the errors got by considering only N terms of the series can be made as small as we please by taking N sufficiently large. Let us consider

$$g_1^* = g'_1 - g_2^*$$

where

$$\begin{aligned} g_2^* &= \frac{xu^*+yv^*}{R} + U^* \\ u^* &= \sum_{n=1}^{N} J_n\,\rho^n \cos n\phi \\ v^* &= \sum_{n=1}^{N} J_n\,\rho^n \sin n\phi \\ U^* &= L_0 + \sum_{n=1}^{N-1} L_n\,\rho^n \cos n\phi \end{aligned} \tag{46}$$

g'_1 is given by (10) and J_n, L_n $(n \geqslant 0)$ by (24-28).

The function g_1^* then satisfies all the requirements of the function g_1 except that g_1^* and $\partial g_1^*/\partial\rho$ do not vanish when $\rho=a$. Hence it is enough to show that $(g_1^*)_{\rho=a}$, $(\partial g_1^*/\partial\rho)_{\rho=a}$ can each be made as small as we please by taking N sufficiently large.

Clearly

$$(g_1^*)_{\rho=a} = \sum_{N}^{\infty} A_n \cos n\phi - J_{N+1}\,a^{N+2}\cos N\phi - mJ_{N+4}\,a^{N+8}\cos N\phi \tag{47}$$

and

$$\left(\frac{\rho\partial g_1^*}{\partial\rho}\right)_{\rho=a} = a\sum_{N}^{\infty} A'_n \cos n\phi - (N+2)\,J_{N+1}\,a^{N+2}\cos N\phi - $$

$$-m(N+8)\,J_{N+4}\,a^{N+6}\cos N\phi. \tag{48}$$

Now $\sum_{N}^{\infty} A_n \cos n\phi$ and $\sum_{N}^{\infty} A'_n \cos n\phi$ are r mainders after N terms of the convergent series (11) for $(g'_1)_{\rho=a}$ and $(\partial g'_1/\partial\rho)_{\rho=a}$.

Therefore, $\sum_{N}^{\infty} A_n \cos n\phi \to 0$

and $\sum_{N}^{\infty} A'_n \cos n\phi \to 0 \quad \text{as} \quad N \to \infty.$ (49)

While $a^{N+2} \to 0$ and $a^{N+\infty} \to 0$ as $a < 1$.

Thus from (47), (48) and (49) we see that $(g_1^*)_{\rho=a}$ and $(\rho \partial g_1^* / \partial \rho)_{\rho=a}$ can be made as small as we please by taking N sufficiently large and desired result is established for g_1. It can similarly be established for g_2.

5. Special Case

If we take $m=0$, we get a circular elastic plate, In this case the solution of the equation is direct since the unknown coefficients which are multiplied by m go out from each of the set of equations (19) - (23). P_n and $(\partial P_n/\partial \rho)$ $(n = 0, 1, 2, \ldots)$ are at once given by (7*a*) - (7*f*) on putting $m=0$, $\lambda = 0$, $R = 1$. The unknown coefficients are given by

$$J_1 = \frac{A'_0}{2a} = 2 \log a + 1 + \frac{c^2}{a^2}$$

$$L_0 = \frac{2A_0 - aA'_0}{2} = 2c^2 \log a - a^2 + c^2$$

$$J_2 = \frac{aA'_1 - A_1}{2a^3} = -\frac{2c}{a^2} + \frac{c^3}{a^4}$$

$$L_1 = \frac{3A_1 - aA'_1}{2a} = -4c \log a - \frac{2c^3}{a^2}$$

and for $n \geqslant 2$,

$$J_{n+1} = \frac{aA'_n - nA_n}{2a^{n+2}} = 2\left(\frac{1}{n+1} \frac{c^{n+2}}{a^{2n+2}} - \frac{1}{n} \frac{c^n}{a^{2n}}\right)$$

$$L_n = \frac{(n+2) A_n - aA'_n}{2a^n} = 2\left(\frac{1}{n-1} \frac{c^n}{a^{2n-2}} - \frac{1}{n} \frac{c^{n+2}}{a^{2n}}\right).$$

These results are the same as obtained by Dube (1971). From these it has been shown that

$$w = \frac{F}{8\pi D}\left[-R^2 \log \frac{cR'}{aR} + \tfrac{1}{2}\left(\frac{c^2}{a^2} R'^2 - R^2\right)\right]$$

which agrees with the results already obtained (Love 1952).

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