A NOTE ON THE RADIAL VIBRATIONS OF A SPHERE OF POROELASTIC MATERIAL

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The frequency equations for the free radial vibrations of a sphere and a spherical shell of porous elastic material containing fluid are obtained here in the low frequency range where the effect of the dissipation due to the flow of fluid has been taken into account. It is seen that the frequency equations involve two (complex) parameters corresponding to the kinds of (complex) dilatational wave velocities. In particular, when the effect of the dissipation is neglected, these parameters become real. The frequency equations for the classical elastic case are recovered from those given here when the fluid is absent. Finally, the first few roots of the frequency equation for the sphere in the non-dissipative case are evaluated numerically.

1. Introduction

The dynamical theory of poroelasticity was established by Biot (1956) and was applied by Deresiewicz and his co-workers in a series of papers (1960, 1961, 1962) to analyse the effect of boundaries on the wave propagation in the fluid filled porous media. Some other problems considered by several authors are listed in a review article by Paria (1966).

The present note is concerned with the free radial vibrations of a sphere and a spherical shell of fluid filled porous elastic material. The expressions for the radial solid and fluid displacements are first obtained in a straight forward way and the frequency equation for the free radial vibration of the sphere is then obtained. The frequency equation for the free vibration of the spherical shell is obtained in a similar way. It is seen that the parameters which occur in the frequency equations are complex in contrast with the non-dissipative case, where these are all real. These parameters depend on the frequency of the vibration and the complex dilatational wave velocities of the first and second kind. In the absence of fluid we recover the frequency equations for the classical case (Love 1952). The frequency equation for the free radial vibrations of the sphere in the non-dissipative case has been solved numerically and the first three roots of this equation have been shown. The results derived here may be relevant in connection with the long period oscillations of the earth, supposed to be a liquid-filled porous solid.

2. Basic Equations : Radial Symmetry

The starting point of our analysis is Biot's equations of dynamical poroelasticity in which we have taken into account the dissipation due to the flow of fluid relative to the solid but restricted to the low frequency range.

The equations of motion are (Deresiewicz and Rice 1962)

$$N\nabla^2\vec{u}+\text{grad}\,[\,(D+N)\,\text{div}\,\vec{u}+Q\,\text{div}\,\vec{U}]$$

$$=\frac{\partial^2}{\partial t^2}[\rho_{11}\,\vec{u}+\rho_{12}\,\vec{U}\,]+b_0\frac{\partial}{\partial t}\,[\vec{u}-\vec{U}] \tag{1}$$

$$\text{grad}\,\,[Q\,\text{div}\,\vec{u}+R\text{div}\,\vec{U}]=\frac{\partial^2}{\partial t^2}\,[\rho_{12}\,\vec{u}\,+\rho_{22}\vec{U}]-b_0\frac{\partial}{\partial t}\,[\vec{u}-\vec{U}] \tag{2}$$

where $\vec{u}$ and $\vec{U}$ are the displacement vector of the solid and fluid respectively and D, N, Q and R (all non-negative) are elastic moduli, ρ_{11}, ρ_{12} and ρ_{22} are dynamical coefficients. Also, b_0, dissipation coefficient related to Darcy's coefficient of permeability k by

$$b_0=\frac{\mu\beta^2}{k}$$

μ is the fiuid viscosity and β, the porosity.

Assuming harmonic time dependence of displacements, we may write

$$\vec{u}=[u\,(x_i),\,v\,(x_i),\,\omega\,(x_i))\,e^{i\omega t}$$

and (3)

$$\vec{U}=[U\,(x_i),\,V\,(x_i),\,W\,(x_i)\,]\,e^{i\omega t}$$

where i=1, 2, 3 when it occurs as a suffix and ω is the frequency of the vibration.

With the aid of (3), equations. (1) and (2) may be written in the form,

$$P\,\nabla^2 e\,+\,Q\,\nabla^2\,\epsilon=\,-\,\omega^2\,[\,\lambda_{11}\,e+\lambda_{12}\,\epsilon\,] \tag{4}$$

$$Q\,\nabla^2\,e\,+\,R\,\nabla^2\,\epsilon=\,-\omega^2\,[\,\lambda_{12}\,e+\lambda_{22}\epsilon\,] \tag{5}$$

where

$$P=D+2N,\ \lambda_{11}=\rho_{11}-\frac{ib_0}{\omega},\ \lambda_{12}=\rho_{12}+\frac{ib_0}{\omega}\ \lambda_{22}=\rho_{22}-\frac{ib_0}{\omega}$$

and e and ϵ are the solid and fluid dilatation respectively (without the harmonic time factor).

Introducing spherical polar coordinate (r, θ, ϕ) and assuming $u_r=u(r)=u$, $u_\theta=0$ and $u_\phi=0$, the strains, e_{ij} of the solid and strains ϵ_{ij}, of fluid can be written in the form

$$\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r},\ e_{\theta\theta}=e_{\phi\phi}=\frac{u}{r},\ e_{r\theta}=e_{r\phi}=e_{\theta\phi}=0 \\ \epsilon_{rr} &= \frac{\partial U}{\partial r},\ \epsilon_{\theta\theta}=\epsilon_{\phi\phi}=\frac{U}{r},\ \epsilon_{r\theta}=\epsilon_{r\phi}=\epsilon_{\theta\phi}=0 \\ e &= \frac{\partial u}{\partial r}+2\frac{u}{r},\ \epsilon=\frac{\partial U}{\partial r}+2\frac{U}{r} \end{aligned} \tag{6}$$

The radial stress and the circumferential stress σ_{rr} and $\sigma_{\theta\theta}$ of the solid and the fluid pressure S, can be written in the form

$$\begin{aligned} \sigma_{rr} &= (D+2N)\frac{\partial u}{\partial r}=\frac{2D}{r}u+Q\left[\frac{\partial U}{\partial r}+2\frac{U}{r}\right] \\ \sigma_{\theta\theta} &= D\frac{\partial u}{\partial r}+2(N+D)\frac{u}{r},+Q\left[\frac{\partial U}{\partial r}+2\frac{U}{r}\right] \\ \zeta &= Q\left[\frac{\partial u}{\partial r}+2\frac{u}{r}\right]+R\left[\frac{\partial U}{\partial r}+2\frac{U}{r}\right]. \end{aligned} \tag{7}$$

The equations (4) and (5) can now be written in the forms

$$\left.\begin{aligned} -\omega^2\left[\lambda_{11}e+\lambda_{12}\epsilon\right] &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(Pe+Q\epsilon\right)\right] \\ -\omega^2\left[\lambda_{12}e+\lambda_{22}\epsilon\right] &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(Qe+R\epsilon\right)\right]. \end{aligned}\right\} \tag{8}$$

Problem I : Free Radial Vibrations of a Sphere : Frequency Equation

We consider a sphere of porous material of radius 'a' with its centre at the origin. We seek solutions of eqns. (8) in the form

$$e=c_1\,e(r) \text{ and } \epsilon=c_2\,\epsilon(r)$$

where the functions $e(r)$ $\epsilon(r)$ must be finite at the origin. The boundary $r=a$ of the sphere is supposed to be free of traction and fluid pressure. Since we are concerned with the free radial vibrations, the boundary conditions are given by

$$\left.\begin{aligned} \sigma_{rr}&=0 \\ S&=0 \end{aligned}\right|\ \text{on } r=a. \tag{9}$$

To solve system of equations (8) we assume

$$\epsilon = C \frac{\text{Sin }(hr)}{r} \tag{10}$$

where C is an arbitray constant and h is a parameter which will be shown to be frequency dependent. Again from the pair of eqns. (8) we deduce

$$e = F \frac{1}{r^2} \frac{d}{dr}\left[r^2 \frac{d\epsilon}{dr}\right] + G\epsilon \tag{11}$$

with

$$\omega^2 F (Q \lambda_{11} - P \lambda_{12} = PR - Q^2$$

$$G (Q \lambda_{11} - P \lambda_{12}) = P\lambda_{22} - Q \lambda_{12} .$$

The equation (11) then leads to

$$e = C (G - h^2 F) \frac{\sin (hr)}{r} \tag{12}$$

where h satisfy the equation

$$h^4 - h^2 \frac{P\lambda_{22} - 2Q\lambda_{12} + R\lambda_{11}}{PR - Q^2} \omega^2 + \omega^4 \frac{\lambda_{11}\lambda_{22} - \lambda_{12}^2}{PR - Q^2} = 0$$

or, more concisely,

$$\left(h^2 - \frac{\omega^2}{v_1^2}\right)\left(h^2 - \frac{\omega^2}{v_2^2}\right) = 0 \tag{13}$$

where V_1^2 and V_2^2 are the squares of the complex velocities of the dilatational wave of the first and second kind respectively (Deresiewicz and Rice 1962). The physical significance of the complex velocities V_1 and V_2 is that their real parts represent the phase velocities of the two kinds of dilatational waves. However, in the non-dissipative case ($b_0 = 0$), the complex velocities reduce to real expressions giving phase velocities.

Now the appropriate solutions of (10) and (11) which are finite for $r \to 0$ may be written as

$$\epsilon = \sum_{i=1,2} C_i \frac{\sin (h_i r)}{r} \tag{14}$$

$$e = \sum_{i=1,2} C_i a_i \frac{\sin (h_i r)}{r} \tag{15}$$

and

$$u = \sum_{i=1,2} C_i a_i \left[\frac{\sin (h_i r)}{h_i^2 r^2} - \frac{\cos (h_i r)}{h_i r}\right] \tag{16}$$

$$U = \sum_{i=1,2} C_i \left[\frac{\sin (h_i r)}{h_i^2 r^2} - \frac{\cos (h_i r)}{h_i}\right] \tag{17}$$

with $a_i = G - h_i^2 F, \quad i = 1, 2$

Hence from the boundary conditions (9) and the equations (14), (15), (16) and (17) and eliminating constants C_i $(i = 1, 2)$ we have the frequency equation of the free radial vibration of a sphere of poroelastic material as

$$\begin{vmatrix} a_1 h_2^2 C_{11} & a_2 h_1^2 C_{12} \\ C_{21} & C_{22} \end{vmatrix} = 0 \tag{18}$$

where

$$C_{1i} = \left[4Nh_i a \cos(h_i a) + \left(Ph_i^2 a^2 - 4N + \frac{Q h_i^2 a^2}{a_i}\right) \sin(h_i a)\right]$$

$$C_{2i} = (Qa_i + R) \sin(h_i a) \quad (i = 1, 2). \tag{18a}$$

If we consider the non-dissipative case ($b_0 = 0$), we get the frequency equation in the same form as equation (18). But the parameters h_i, a_i involved in the frequency equation are all real ; and in particular, the h_i s are related to the phase velocities c_1, c_2 of the dilatational waves of the first and second kind respectively by

$$h_i^2 = \frac{\omega^2}{c_i^2} \quad (i = 1, 2). \tag{19}$$

Moreover if we make $Q \to 0$, $R \to 0$, $\rho_{12} \to 0$, $\rho_{22} \to 0$ such that $Q^2/R \to 0$ and $\frac{H}{\rho} \frac{\rho_{11}\rho_{22} - \rho_{12}^2}{\rho_{11}R + \rho_{22}P - 2\rho_{12}Q} \to 1$ which imply $c_2 \to 0$, $c_1^2 \to \frac{D+2N}{\rho_{11}}$ and $h_2 a \to \infty$, we have the frequency equation as

$$\frac{\tan \omega'}{\omega'} = \frac{1}{1 - \gamma \omega'^2} \tag{20}$$

where

$$\gamma = \frac{P}{4N}, \quad \omega^1 = h_1 a .$$

The equation (20), as expected, is in agreement with the classical result given in Love (1952).

Problem II : Free Radial Vibration of Spherical Shell : Frequency Equation

We consider now the free radial vibration of a spherical shell bounded by surfaces $r = a$ and $r = b$ $(b > a)$. We suppose that the inner surface of the spherical shell rests on a permeable, solid core and that the outer surface is also fully permeable. The appropriate boundary conditions are then

$$\left.\begin{aligned} \sigma_{rr} &= 0 \\ S &= 0 \end{aligned}\right| \text{ on } r = a \text{ and } r = b. \tag{21}$$

To solve the system of equations (8), we assume first,

$$\epsilon = \frac{A \sin(hr) + B \cos(hr)}{r} \tag{22}$$

and hence

$$e = (G - h^2 F)\,\epsilon. \tag{23}$$

Then proceeding as earlier, we may have the appropriate solution of (8) as

$$\epsilon = \sum_{i=1,2}\left[A_i \frac{\sin (h_i r)}{r} + B_i \frac{\cos (h_i r)}{r}\right] \tag{24}$$

$$e = \sum_{i=1,2} a_i\left[A_i \frac{\sin (h_i r)}{r} + B_i \frac{\cos (h_i r)}{r}\right] \tag{25}$$

and

$$u = \sum_{i=1,2} a_i\left[A^i\left\{\frac{\sin (h_i r)}{h_i^2\, r^2} - \frac{\cos (h_i r)}{h_i r}\right\} + B_i\left\{\frac{\sin (h_i r)}{h_i r} + \frac{\cos (h_i r)}{h_i^2\, r^2}\right\}\right] \tag{26}$$

$$U = \sum_{i=1,2}\left[A_i\left\{\frac{\sin (h_i r)}{h_i^2\, r^2} - \frac{\cos (h_i r)}{h_i r}\right\} + B_i\left\{\frac{\sin (h_i r)}{h_i r} + \frac{\cos (h_i r)}{h_i^2\, r^2}\right\}\right] \tag{27}$$

where $a_i = G - h_i^2 F, \quad i = 1, 2$

and the frequency equation may be written in the form

$$\begin{vmatrix} a_1 h_2^2 C'_{11} & a_1 h_2^2 C'_{12} & a_2 h_1^2 C'_{13} & a_2 h_1^2 C'_{14} \\ a_1 h_2^2 C'_{21} & a_1 h_2^2 C'_{22} & a_2 h_1^2 C'_{23} & a_1 h_1^2 C'_{24} \\ C'_{31} & C'_{32} & C'_{33} & C'_{34} \\ C'_{41} & C'_{42} & C'_{43} & C'_{44} \end{vmatrix} = 0 \tag{28}$$

where $C'_{11}, C'_{13}, C'_{31}, C'_{33}$ are same as $C_{11}, C_{12}, C_{21}, C_{22}$ respectively defined in (18*a*),

$$C'_{ij} = \cos (h_k a)\left(P h_k^2 a^2 - 4N + \frac{Q\, h_k^2\, a^2}{a_k}\right) - 4N h_k a \sin (h_k a)$$

$C'_{3j} = (Q a_k + R) \cos (h_k a)$, in which when $j = 2,\ k = 1$ and $j = 4,\ k = 2$. The coefficients $C'_{21}, C'_{22}, C'_{23}, C'_{24}, C'_{41}, C'_{42}, C'_{43}$ and C'_{44} are the same as $C_{11}, C_{12}, C_{13}, C_{14}, C_{31}, C_{32}, C_{33}$ and C_{34} respectively except that in these expressions the quantity a wherever it occurs should be replaced by b. Due to the very intricate nature of the frequency equation (28) for a spherical shell it has not been feasible to give an indication of its roots. However, as a check, we can derive the frequency equation of an elastic spherieal shell by putting in (28) $b_0 = 0$ and then by making use of the limiting conditions stated just below equation (19). The equation so derived coincides with known results (Love 1952).

3. Numerical Analysis of the Frequency Equation for the Sphere

We shall now calculate the first few roots of the frequency equation (for the non-dissipative case) :

$$\begin{vmatrix} a_1\, h_2^2\, c_{11} & a_2\, h_1^2\, c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0 \tag{29}$$

when $h_i = \frac{\omega^2}{c_i^2}$, $i = 1, 2$. The only experimental results which have been so far published being those of Fatt (1959), we take the values of the parameters for a particular material as in Deresiewicz and Rice (1962) as follows :

$$P = 1.4445, \quad Q = 0.1078, \quad R = 0.0473, \quad N = 0.3996, \quad H = 1.7074$$
$$\rho = 2.1372, \quad \sigma_{11} = 0.8460, \quad \sigma_{12} = 0.0631, \quad \sigma_{22} = 0.0277, \quad \gamma_{11} = 0.9012,$$
$$\gamma_{12} = -0.0010 \quad \gamma_{22} = 0.1008. \quad a_1 = 1.14067, \quad a_2 = -0.08923, \frac{h_1}{h_2} = 0.4636.$$
$$\rho_{ij} = \rho\gamma_{ij}, \quad (i, j = 1, 2)$$

Now the frequency equation (29) takes the form

$$\frac{0.0304}{\tan \omega_0} + \frac{0.1855}{\tan (0.4636\omega_0)} = \frac{0.4305}{\omega_0} - 0.0873\omega_0 . \tag{30}$$

The first few roots of eqn. (30) have been found to be

$$\omega_0 = h_2 a = \frac{\omega a}{c_2} = 2.9408, 5.7800, 13.2109.$$

The frequency equation for the spherical shell, though more complicated can be numerically solved in a similar way.

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CORRIGENDA

Integrability Theorems for Dirichlet series with Positive coefficients

by

S. M. MAZHAR *and* R. S. KHAN

Indian J. Pure and Applied Mathematics Vol. 7, 1976, 53—65.

Page No.	Line No.	Read	Instead of
54	4^-	$(1-e^{-t})$	$(1-p^{-t})$
55	3^+	$\frac{\lambda_{nj}}{\lambda_n}$	$\frac{\lambda_{nj}}{\lambda}$
55	4^+	$\sum_{1}^{\infty} \ldots r_n^p$	$\sum_{1}^{\infty} \cdot\cdot \gamma_n^p$
56	4^+	$\left(\frac{p}{p-1}\right)^p$	$\left(\frac{p}{p-1}\right)$
56	9^+	r_n^p	γ_n^p
56	5^-	$o\left(\frac{2^\nu}{\cdot\cdot}\right)$	$0\left(\frac{2^\nu}{\cdots}\right)$
56	2^-	Replace γ by r	—
56	2^-	$\sum_{k=1}^{n}$	$\sum_{k=1}$
56	1^-	$\sum_{r=1}^{s} \frac{2^r(\lambda_{2^{r+1}}-\lambda_{2^r})}{(\lambda_{2^r})^{jp}}$	$\sum_{r=1}^{s} \frac{2^\nu(\lambda_{2^\nu+1}-\lambda_\nu)}{\lambda_{2^\nu}}$
57	...	Replace γ by r throughout	
57	1^+ and 5^+	$\sum_{k=2^{\nu-1}}^{\infty}\cdots$	$\sum_{k=2\nu-1}^{\infty}\cdots$
57	6^+	$\left(\sum_{k=n}^{\infty}\cdots\right)^p\}$	$\left(\sum_{k=n}^{\infty}\cdots\right)\}^p$
57	7^+	$\left(\sum_{k=n}^{s}\cdots\right)^p$	$\left(\sum_{k=}^{\infty}\cdots\right)^p$
58	...	Replace γ by r throughout	

CORRIGENDA

Page No.	Line No.	Read	Instead of
58	5^+	$\sum_{k=1}^{n-1} \cdots$	$\sum_{k=1}^{-1} \cdots$
58	2^-	$\sum_{k=0}^{n} \cdots$	$\sum_{k=1}^{n} \cdots$
59	...	Replace γ by r throughout	
59	1^-	$\sum_{k=n}^{\infty} \cdots$	$\sum_{k=1}^{\infty} \cdots$
60	1^+	$(\lambda_{n+1} - \lambda_n)$	$(\lambda_{n+1} - \lambda)$
60	1^+	$\sum_{k=n}^{\infty} \cdots$	$\sum_{k=2}^{\infty} \cdots$
60	2^+	$\left(1 - e^{-\frac{1}{\lambda_2}}\right)$	$\left(1 - e^{-\frac{1}{\lambda_1}}\right)$
60	4^-	$\sum_{n=1}^{\infty} \cdots$	$\sum^{\infty} \cdots$
60	2^-	$\left(1 - \frac{1}{\lambda_n}\right)$	$\left(1 - \frac{1}{\lambda_\mu}\right)$
60	1^-	$\sum_{n=2}^{\infty} \cdots$	$\sum_{n=1}^{\infty}$
60	5^+	$\int_{1-\frac{1}{\lambda_{n-1}}}^{1-\frac{1}{\lambda_n}}$	$\int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}}$
61	5^+	$\sum_{n=2}^{\infty} \cdots$	$\sum_{n=1}^{\infty} \cdots$
61	9^+	$\left(\sum_{j=1}^{\infty} e^{-\frac{\lambda_{jn}}{\lambda_n}} \cdots\right)$	$\left(\sum_{j=1}^{\infty} e^{-\frac{\lambda_{jn}}{\lambda_1}} \cdots\right)$
61	2^-, 3^-, 4^-, 5^-	$(j+1)n-1$	$(j+1)^{n-1}$
61	4^-	$\sum_{j=1}^{\infty} \cdots$	$\sum_{j+1}^{\infty} \cdots$
62	6^-	$(1-x)^{-2}$	$(1-x)^{-1}$
62	4^-	r_n^p	γ_n^p
63	4^+	$\left(1 - \frac{1}{\lambda_{n+1}}\right)$	$\left(1 - \frac{1}{\lambda_{k+1}}\right)$

CORRIGENDA

63	5+	$(\lambda_{n+1}-\lambda_n)$	$(\lambda_{n+1}-\lambda_u)$
63	6+	r_n^p	γ_n^p
63	7+	a_n^p	γ_n^p
64	5+	$(\lambda_{n+1}-\lambda_n)$	$(\lambda_{n+1}-\lambda_u)$
64	6+	$(\lambda_{n+1}-\lambda_n)$	$(\lambda_{n+1}-\lambda_n$
64	5−	$\sum_{n=2}^{\infty} \cdots$	$\sum_{n=1}^{\infty} \cdots$
65	1+	$\sum_{n=2}^{\infty} \cdots$	$\sum_{n=1}^{\infty} \cdots$
65	8−	$\sum_{4}^{\infty} a_n x^{n^2}$	$\sum_{0}^{\infty} a_n x^{n^2}$
65	4−	$\left(\sum_{k=0}^{n} \cdots\right)$	$\left(\sum_{g=0}^{u} \cdots\right)$