A REPRESENTATION OF ONE-SIDED LAPLACE TRANSFORMABLE GENERALIZED FUNCTIONS

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We prove that every generalized function belonging to distributional one-side Laplace transformation can be represented by a finite sum of derivatives of continuous functions decaying exponentially at infinity.

1. Introduction

The conventional one-sided Laplace transformation is defined by

$$F(s) = \int_0^\infty f(t)\ e^{-st}\ dt$$

where $f(t)$ is suitably restricted conventional function on I: $0 < t < \infty$. Thus this transformation maps $f(t)$ into a function $F(s)$ of the complex variable s.

Let $\mathscr{L}_{+,a}$ denotes the space of all smooth functions $\phi(t)$ on I such that

$$\lambda_{a,k}(\phi) \triangleq \sup_{0<t<\infty} \left| e^{at} D^k \phi(t) \right| < \infty ,$$

$$k = 0, 1, 2, \quad .. \quad .. \quad .. \qquad (1)$$

and its topology is generated by $\{\lambda a,k\}_{k=0}^{\infty}$. $\mathscr{L}_{+,a}$ is a Hausdorff, locally convex, first countable linear space. It is complete and therefore Fre'chet (see Zemanian 1968, p. 90). For each fixed complex number s in the strip $\Omega_f = \{ s : \sigma_f < \text{Re}\ s < \infty \}$, $\mathscr{L}_{+,a}$ contains e^{-st} .

The one-sided Laplace transform $\mathscr{L}_+ f$ of f is defined by

$$F(s) \triangleq (\mathscr{L}_{\times} + f)(s) \triangleq \langle f(t),\ e^{-st} \rangle, \text{Re}\ s > \sigma_f . \quad .. \quad .. \qquad (2)$$

Under these circumstances we call f a one-sided Laplace transformable generalized function and (2) a one-sided Laplace transform.

It is the case that the dual $\mathscr{L}'_{+,a}$ contains all distributions of compact support on $(0, \infty)$. Also, any locally integrable function $f(t)$ on $0 < t < \infty$ and such that

$$\int_0^\infty | f(t)\ e^{-at} |\ dt < \infty \quad .. \quad .. \quad .. \qquad (3)$$

is a member of $L'_{+,a}$.

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We now prove that every generalized function belonging to $\mathscr{L}'_{+,a}$ can be represented by a finite sum of derivatives of continuous functions decaying exponentially at infinity. Our proof is analogous to the method employed in structure theorems for Schwartz distributions (e.g. Edwards, 1965, p. 317 ff.); Treves 1967, p. 272 ff.).

2. Theorem

Let $f \in \mathscr{L}'_{+,a}$ Then f is equal to a finite sum

$$\sum_{i=1}^{k} C_i \left(\frac{d}{dt}\right)^i \left[e^{at} P_i(t) F_i(t) \right] \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where $F_i(t)$ are continuous on $(0, \infty)$ and the $P_i(t)$ are polynomials of degree k.

Proof : For every $f \in \mathscr{L}'_{+,a}$ there exists a positive constant C and a non-negative integer r such that, for all $\phi \in D(I)$.

$$| < f, \phi > | \leqslant C \max_{0 \leqslant k \leqslant r} \lambda_{a,i} k(\phi),$$

$D(I)$ being the space of smooth functions with compact support in $(0, \infty)$. (Zemanian 1968, p. 52 ff.).

That is,

$$| < f, \phi > | \leqslant C \max_{0 \leqslant i \leqslant r} \sup_{0 < t < \infty} | e^{at} D^i \phi(t) | \quad \ldots \quad (5)$$

Let us set

$$\phi_r(t) = e^{at} \phi(t). \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Clearly $\phi_r(t) \in D(I)$. In fact $\phi \to \phi_r$ is a one-to-one linear map of $D(I)$ onto itself.

Then

$$\phi(t) = e^{-at} \phi_r(t).$$

And

$$\frac{d\phi}{dt} = e^{-at}\left(-a + \frac{d}{dt}\right) \phi_r(t).$$

Let supp ϕ = supp ϕ_r = $[A, B]$. Then

$$\left|\frac{d\phi}{dt}\right| \leqq A_{I1} e^{-at} \left(\left|\phi_r\right| + \left|\frac{d\phi_r}{dt}\right| \right),$$

where $A_{il} = \max(-a, 1)$. Continuing in this fashion, we have

$$\left|\left(\frac{d}{dt}\right)^i \phi\right| \leqq A_i\, e^{-at} \sum_{q=0}^{i} \left|\left(\frac{d}{dt}\right)^q \phi_r\,(t)\right| . \qquad (7)$$

Substituting (7) into (5) we have

$$| < f, \phi > | \leqq C' \max_{0 \leqslant i \leqslant r} \sup_{0 < t < \infty} \left| \sum_{q=0}^{i} \left| \left(\frac{d}{dt}\right)^q \phi_r\,(t) \right| \right|$$

$$\leqq C'' \max_{0 \leqslant i \leqslant r} \sup_{0 < t < \infty} \left| \left(\frac{d}{dt}\right)^i \phi_r\,(t) \right|$$

where C' and C'' are obvious constants.

We can write for every $\psi \in D\,(I)$

$$\psi\,(t) = \int_0^t \frac{d\psi}{dx}\,.\,dx\,.$$

Hence

$$\sup_{0 < t < \infty} \left|\psi\,(t)\right| \leqslant \sup_{0 < t < \infty} \int_0^t \left|\frac{d\psi}{dx}\right| dx$$

$$= \left\| \frac{d\psi}{dx} \right\|_{L_1\,(0,\,\infty)} \qquad (9)$$

where $L_1\,(0, \infty)$ is the space of equivalence classes of Lebesgue integrable functions on $(0, \infty)$ whose topology is defined by the norm

$$\| f \|_{L_1\,(0,\,\infty)} = \int_0^\infty | f |\; dx < \infty\,,\; f \in L_1\,(0, \infty)\,.$$

The bound (9) enables us to write (8) as

$$| < f, \phi > | \leqslant C''' \max_{1 \leqslant i \leqslant r+1} \left\| \left(\frac{d}{dt}\right)^i \phi_r\,(t) \right\|_{L_1\,(0,\,\infty)}. \qquad (10)$$

Since $D(I)$ is a linear manifold of $L_1(0, \infty)$, (10) states that the linear functional f is continuous on $MD(I)$ (i.e $D(I)$) for the topology induced on it by $L_1\,(0, \infty)$, where

$$M : \phi \to \left(\frac{d}{dt}\right)^i \phi \quad \Big|_{1 \leqslant i \leqslant r+1}$$

is the linear one-to-one mapping of $D(I)$ into $L_1(0, \infty)$. Hence by Hahn-Banach theorem, f can be extended as a continuous linear functional in the whole of $L_1(0, \infty)$. But the dual of $L_1(0, \infty)$ is isomorphic with $L_{\infty(0,\infty)}$, the space of all equivalence classes of complex-valued integrable functions on $(0, \infty)$ such that for every $f \in L_{\infty(0,\infty)}$, there exists an M such that $|f| \leqslant M$ a.e. Therefore there are functions $g_i \in L_{\infty(0,\infty)}$, $1 \leqslant i \leqslant r+1$, such that

$$< f, \phi > = \sum_{i=1}^{r+1} < g_i, \left(\frac{d}{dt}\right)^i \phi_\gamma (t) > .$$

Recalling (6) we have

$$< f, \phi > = \sum_{i=1}^{r+1} < (-1)^i \left(\frac{d}{dt}\right)^i g_i , e^{at} \phi (t) > .$$

Therefore

$$f = \sum_{i=1}^{r+1} e^{at} (-1)^i \left(\frac{d}{dt}\right)^i g_i . \qquad \qquad (11)$$

For each i, we set

$$h_i (t) = (-1)^i \int_0^t g_i (x)\, dx.$$

Since $g_i \in L_{\infty(0,\infty)}$, the functions h_i are continuous on $(0, \infty)$ and

$$\left| h_i \right| \leqq \int_0^\infty \left| g_i \right| dx \leqq \left| t \right| \max_{0 < t < \infty} \left| g_i \right|$$

$$= | t | \; \| g_i \|_{L_\infty (0,\infty)} .$$

Further more

$$g_i = (-1)^i \left(\frac{d}{dt}\right)^i h_i .$$

Hence

$$f = \sum_{i=2}^{r+2} e^{at} \left(\frac{d}{dt}\right)^i h_i . \qquad \qquad (12)$$

By letting $r + 2 = k$ and using the differentiation formulae

$$u (t) \left(\frac{d}{dt}\right)^i h_i = \sum_{j=0}^{i} (-1)^i \binom{i}{j} \left[u^{(j)} h_i \right]^{(i-j)}$$

and $$(ab)^{(j)} = \sum_{q=0}^{j} \binom{i}{q} \overset{(j-q)}{a} \overset{(q)}{b}$$

we can write (12) as in (4) where the F are continuous functions of h_l and are therefore continuous functions on $(0, \infty)$. This proves the theorem.

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