ELECTROMAGNETIC FIELDS CONFORMAL TO SOME NON-EMPTY SPACE-TIMES

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In this paper we investigate the existence of electromagnetic fields conformal to some well-known non-empty space-times. A static non-null electromagnetic field conformal to de Sitter universe has been obtained which is uniform and reduces to a wrench. It is shown that in an electromagnetic field conformal to the field of 'total radiation' the conformal distortion effect is propagated with fundamental velocity.

1. Introduction

The conformal curvature tensor plays an important role in the theory of gravitational radiation in general relativity. Two Riemannian spaces V_n and $\bar{V}_n$ are said to be conformal to each other if their metric tensors g_{ij} and $\bar{g}_{ij}$ satisfy the equation

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1.1)$$

where σ is any scalar function of coordinates (Eisenhart 1949). By virtue of the the equality of conformal curvature tensors, two gravitational space-times which are conformal to each other will have similar radiative propery. Therefore, in the general theory of relativity a study of space-times which are conformal to those of well known radiative properties is of special interest.

The relation between Ricci tensors of V_n and $\bar{V}_n$ spaces is

$$\bar{R}_{ij} = R_{ij} + 2\,(\sigma_{;\,ij} - \sigma_{;\,i}\,\sigma_{;\,j}) + g_{ij}\,g^{rs}\,(\sigma_{;rs} + 2\sigma_{;\,r}\,\sigma_{;\,s}) \qquad \ldots \qquad \ldots \qquad (1.2)$$

where semicolon denotes covariant differentiation. Here we take $n = 4$.

In Einstein-Maxwell's theory of electromagnetism we write the field equations in terms of the energy tensor which enable us to derive the field. Since the energy tensor is the source of the gravitational field, a completely geometric description can be given for a gravitational field produced by a source free electromagnetic field. If the space-time $\bar{V}_4$ represents non-null electromagnetic field the Rainich Equations of the already unified theory (Witten 1962)

$$\bar{R} = 0$$

$$\bar{R}_{ij}\,\bar{R}^{jk} = \tfrac{1}{4}\,\bar{R}_{mn}\,\bar{R}^{mn}\,\delta_i^k$$

and

$$\theta_{l;m} - \theta_{m;l} = 0 \qquad .. \qquad .. \qquad .. \qquad (1.3)$$

where

$$\theta_l = \frac{\sqrt{-g}\, \epsilon_{lmnp}\, \bar{g}^{nr}\, \bar{R}^m_{s;r}\, \bar{R}^{sp}}{\bar{R}^k_h\, \bar{R}^h_k}$$

must be satisfied. If the field be null we must have

$$\bar{R}_{ij}\, \bar{R}^{jk} = 0. \qquad .. \qquad .. \qquad .. \qquad (1.4)$$

A class of solutions representing electromagnetic fields conformal to some empty space-times has been obtained by Singh and Roy (1966). In this paper an attempt is made to investigate whether the existence of electromagnetic fields conformal to some well known non-empty space-times is possible. In section 2, we obtain the static electromagnetic field conformal to de Sitter universe which is an Einstein space. The field is found to be uniform and reduces to a wrench in the given coordinate system and is of Petrove type *ID*. In section 3, it is found that in the electromagnetic field conformal to the field of 'total radiation' the conformal distortion factor is propagated with the fundamental velocity. It is also found that electromagnetic field conformal to Krishna Rao-field of 'total radiation' does not exist in Einstein's theory of gravitation. The necessary condition for the existence of conformal electromagnetic field is that the field itself be flat and thus we arrive at a well known result obtained by Singh and Roy (1966).

2. Electromagnetic Field Conformal to Einstein Space

If V_4 is an Einstein space we have

$$R_{ij} = Kg_{ij}, \qquad K \equiv R/4. \qquad .. \qquad .. \qquad .. \qquad (2{\cdot}1)$$

Substituting from (2.1) in (1.2) we get

$$\bar{R}_{ij} = Kg_{ij} + 2\,(\sigma_{;ij} - \sigma_{;i}\,\sigma_{;j}) + g_{ij}\, g^{rs}\,(\sigma_{;rs} + 2\sigma_{;r}\,\sigma_{;s}) \qquad .. \qquad .. \qquad (2{\cdot}2)$$

and so

$$R = 4Ke^{-2\sigma} + 6\, e^{-2\sigma}\, g^{ij}\,(\sigma_{;ij} + \sigma_{;i}\,\sigma_{;j}). \qquad .. \qquad .. \qquad .. \qquad (2{\cdot}3)$$

For vanishing R we have

$$g^{ij}\,(\sigma_{;ij} + \sigma_{;i}\,\sigma_{;j}) = -\tfrac{2}{3}\, K. \qquad .. \qquad .. \qquad .. \qquad (2{\cdot}4)$$

The equation (2.3) may be written in the form

$$(g^{ij}\, e^{\sigma})_{;ij} = -\tfrac{2}{3}\, Ke^{\sigma}. \qquad .. \qquad .. \qquad .. \qquad (2{\cdot}5)$$

The equation (2.5) is a non-homogeneous wave equation in curved space-time. In empty space-time gravitational disturbances move with the fundamental velocity and it is well known that a gravitational wave slows down in material medium due to non-homogeneity of the field equations. The non-homogeneity of the equation (2·5) appears to be due to the non-emptiness of the V_4.

From eqns. (2.2) and (2.4) we have

$$\bar{R}_{ij} = 2\,(\sigma_{;ij} - \sigma_{;i}\,\sigma_{;j}) - \tfrac{1}{2}\, g_{ij}\, g^{rs}\,(\sigma_{;rs} - \sigma_{;r}\,\sigma_{;s}).$$

For the electromagnetic field to be null we require that

$$\sigma_{;ij} = \sigma_{;i}\,\sigma_{;j} \qquad \ldots \quad \ldots \quad \ldots \qquad (2{\cdot}6)$$

But under this differential condition $\bar{R}_{ij} = 0$. Hence under the condition (2.6) the space $\bar{V}_4$ cannot represent null electromagnetic field. The necessary condition that spaces conformal to Einstein spaces be empty is given by (2.6).

We consider the space-time conformal to de Sitter cosmological universe (Tolman 1966), viz.

$$ds^2 = e^{2\sigma}\left[-\frac{dr^2}{1-r^2/R^2} - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 + (1 - r^2/R^2)\, dt^2\right] \qquad (2{\cdot}7)$$

where $\sigma = \sigma\,(r)$ only.

For the line-element (2.7) the non-vanishing components of Ricci tensor are

$$\bar{R}^1_1 = -\,e^{-2\sigma}\left[3(1 - r^2/R^2)\,\sigma_{11} + \left\{\frac{2}{r}\,(1 - r^2/R^2) - \frac{4r}{R^2}\right\}\sigma_1 - \frac{3}{R^2}\right]$$

$$\bar{R}^2_2 = \bar{R}^3_3 = -e^{-2\sigma}\left[(1 - r^2/R^2)\,\sigma_{11} + 2(1 - r^2/R^2)\,\sigma_1^2 + \left\{\frac{4}{r}\,(1 - r^2/R^2) - \frac{2r}{R^2}\right\}\sigma_1 - \frac{3}{R^2}\right]$$

$$\bar{R}^4_4 = -\,e^{-2\sigma}\left[(1 - r^2/R^2)\,\sigma_{11} + 2\,(1 - r^2/R^2)\,\sigma_1^2 + \left\{\frac{2}{r}\,(1 - r^2/R^2) - \frac{4r}{R^2}\right\} - \frac{3}{R^2}\right] \qquad (2{\cdot}8)$$

and the scalar curvature R is given by

$$\bar{R} = -\,6e^{-2\sigma}\left[(1 - r^2/R^2)\,\sigma_{11} + (1 - r^2/R^2)\,\sigma_1^2 + \left\{\frac{2}{r}\,(1 - r^2/R^2) - \frac{2r}{R^2}\right\}\sigma_1 - \frac{2}{R^2}\right] \quad \ldots \quad \ldots \qquad (2{\cdot}9)$$

where

$$\sigma_i = \frac{d\,\sigma}{dx^i} \quad (i = 1).$$

From the Rainich algebraic equation (1·3) three possible cases arise (Witten 1962) :

Case 1

(i) $\bar{R}^2_2 = -\bar{R}^3_3$, (ii) $\bar{R}^4_4 = -\bar{R}^1_1$, (iii) $\bar{R}^4_4 = \bar{R}^2_2$.

Case 2

(i) $\bar{R}^2_2 = -\bar{R}^3_3$, (ii) $\bar{R}^4_4 = -\bar{R}^1_1$, (iii) $\bar{R}^4_4 = -\bar{R}^2_2$.

Case 3

(i) $\bar{R}^2_2 = \bar{R}^3_3$, (ii) $\bar{R}^4_4 = \bar{R}^1_1$, (iii) $\bar{R}^4_4 = -\bar{R}^2_2$. .. (2·10)

It can easily be seen that in first two cases all components of the Ricci tensor vanish. Therefore we discuss here only the third case. The first condition in Case 3 is identically satisfied.

The second relation gives

$$\sigma_{11} - \sigma_1^2 = 0. \qquad \text{.. .. ..} \quad (2\cdot11)$$

Also the relation (iii) gives

$$(1 - r^2/R^2)\,\sigma_{11} + 2(1 - r^2/R^2)\,\sigma_1^2 + 3\left(\frac{1}{r} - \frac{2r}{R^2}\right)\sigma_1 - \frac{3}{R^2} = 0. \qquad (2\cdot12)$$

Using eqn. (2.11) in (2.12) we have

$$(1 - r^2/R^2)\,\sigma_1^2 + \left(\frac{1}{r} - \frac{2r}{R^2}\right)\sigma_1 - \frac{1}{R^2} = 0. \qquad \text{.. ..} \quad (2\cdot13)$$

Solving equation (2.13) for σ_1 we obtain

$$\sigma_1 = \left\{-\left(\frac{1}{r} - \frac{2r}{R^2}\right) \pm \frac{1}{r}\right\} \Big/ 2\,(1 - r^2/R^2)\,.$$

Taking positive sign and integrating we have

$$e^{2\sigma} = \frac{a}{1 - r^2/R^2}\,, \qquad a = \text{const.}$$

But this value of σ does not satisfy the equations (2.11), (2.12) and (2.13). Taking the negative sign we have

$$\sigma_1 = -\frac{1}{r}$$

and therefore

$$e^{2\sigma} = \frac{a}{r^2}\,, \qquad a = \text{const.} \qquad \text{.. .. ..} \quad (2\cdot14)$$

This value of σ satisfies eqns. (2·11), (2·12) and (2·13) and also $\bar{R} = 0$.

Thus the line-element representing electromagnetic field conformal to de Sitter model is

$$ds^2 = \frac{a}{r^2}\left[-\frac{dr^2}{1 - r^2/R^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + (1 - r^2/R^2)dt^2\right] \qquad (2\cdot15)$$

The metric (2.15) has singularity at $r = 0$. So that the charge is situated at the origin of coordinates. At $r = \infty$ the space-time is annihilated.

The nonvanishing components of electromagnetic field tensor F_{ij} are

$$F_{14} = \frac{A}{r^2}, \quad F_{23} = B \sin\theta \qquad \ldots \quad \ldots \quad \ldots \quad (2\cdot16)$$

where

$$A^2 + B^2 = a^2/4\pi.$$

The electromagnetic field thus reduces to a wrench in the given coordinate system. Also it is easy to see that $F^{ij}{}_{;j} = 0$ which implies that the electromagnetic field is uniform.

If we define

$$*F_{ij} = \tfrac{1}{2}\,\epsilon^{ijkl}\,F_{kl},$$

where ϵ^{ijkl} is Levi-Civita tensor, we have from (2.16) that

$$*F^{14} = F_{23}, \qquad *F^{23} = F_{14}$$

and thus we find that

$$F_{ij} \; F^{ij} = \frac{2}{a^4} B^2 - \frac{2r^2}{a^4} A^2 \neq 0$$

and

$$F_{ij} *F^{ij} = \frac{2AB\sin\theta}{r^2} \neq 0.$$

Thus this electromagnetic field is non-null. It has been shown by Singh and Roy [1972] that non-null electromagnetic fields correspond to Petrow type *ID*.

3. Electromagnetic Field Conformal to the Field of "Total Radiation"

Suppose V_4 is a field of 'total radiation' in the sense of Lichnerowicz (1960). Then

$$R_{ij} = \nu k_i k_j \qquad \ldots \quad \ldots \quad \ldots \quad (3\cdot1)$$

where ν is a scalar and k_i is null vecotr field $(k_i k^i = 0)$. Lichnerowicz defines this as a state of 'total radiation' and for pure gravitational radiation we must have $\nu = 0$.

Let $\bar{V}_4$ be a Riemannian fourfold which is conformal to V_4. We have form (3·1) and (1·2) :

$$\bar{R}_{ij} = \nu k_i k_j + 2(\sigma_{;ij} - \sigma_{;i}\,\sigma_{ij}) + g_{ij}\, g^{rs}\,(\sigma_{;rs} + 2\sigma_{;r}\,\sigma_{;s}). \quad \text{(3·2)}$$

Therefore

$$\bar{K} = 6\, \bar{e}^{2\sigma}\, g^{ij}\,(\,\sigma_{;ij} + \sigma_{;i}\,\sigma_{;j}\,)\,. \quad \text{(3·3)}$$

For vanishing $\bar{R}$ (3.3) gives

$$g^{ij}\,(\,\sigma_{:ij} + \sigma_{;i}\,\sigma_{ij}\,) = 0. \quad \text{(3·4)}$$

This equation may be written in the form

$$(\,g^{ij}\, e^{\sigma}\,)_{;ij} = 0. \quad \text{(3.5)}$$

The equation (3.5) has been obtained by Singh and Roy (1966) in the case when V_4 is empty. It is shown here that in the case of 'total radiation' also the same condition holds. It follows that e^{σ} propagates with the fundamental velocity.

A class of exact wave solutions of the Einstein-Rosen cylindrically symmetric space-time corresponding to the field equations of Lichnerowicz's 'total radiation' has been obtained by Krishna Rao (1964). A particular solution in cylindrical polar coordinates (r, ϕ, z, t) is

$$ds^2 = e^{2f}\,(dt^2 - dr^2) - r^2 d\phi^2 - dz^2, \quad \text{(3·6)}$$

where f is a function of $r - t$. The metric (3.6) having conformal curvature tensor of Petrov type N, represents gravitational radiation in non-empty space-times.

For the metric (3.6) the equation (3.5) gives

$$\frac{\partial^2 e^{\sigma}}{\partial r^2} - \frac{\partial^2 e^{\sigma}}{\partial t^2} + \frac{1}{r}\frac{\partial e^{\sigma}}{\partial r} = 0\,, \quad \text{(3·7)}$$

when σ is a function of r and t. Solutions of the equation (3.7) are well known

$$e^{\sigma} = A\, J_0\,(wr)\cos wt + B\, N_0\,(wr)\sin wt,$$

where J_0 and N_0 are Bessel's functions.

The conformal form of the metric (3·6) can be written as

$$ds^2 = e^{2\sigma}\,[\,e^{2f}\,(dt^2 - dr^2) - r^2 d\phi^2 - dz^2\,]\,. \quad \text{(3·8)}$$

It is found that the corresponding conformal curvature tensor also belongs to the Petrov type N and therefore can be interpreted as material distributions pervaded by gravitational radiation. It has already been shown by Krishna Rao and Vaidya (1966) that when σ is a function of t alone, the metric (3·8) represents for certain values of σ, the Einstein—de Sitter and steady state cosmological models pervaded by type N gravitational radiation.

Here we suppose that $\sigma = \sigma\ (r, t)$ for the metric (3·8) the nonvanishing components of Ricci tensor are

$$\bar{R}^1_1 = -e^{-2(\sigma+f)} \left[3\sigma_{11} - \sigma_{44} + 2\sigma_4 f' - 2\sigma_1 f' - 2\sigma_4^2 - \frac{\sigma_1}{r} - \frac{f'}{r} \right],$$

$$\bar{R}^2_2 = -e^{-2(\alpha+f)} \left[\sigma_{11} - \sigma_{44} + 2\sigma_1^2 - 2\sigma_4^2 + \frac{3\sigma_1}{r} \right],$$

$$\bar{R}^3_3 = -e^{-2(\sigma+f)} \left[(\sigma_{11} - \sigma_{44} + 2\sigma_1{}^2 + 2\sigma_4{}^2 + \frac{\sigma_1}{r} \right],$$

$$\bar{R}^4_4 = -e^{-2(\sigma+f)} \left[\sigma_{11} - 3\sigma_{44} + 2\sigma_1{}^2 + 2\sigma_1 f' - 2\sigma_4 f' + \frac{\sigma_1}{r} + \frac{f'}{r} \right],$$

$$\bar{R}^4_1 = -\bar{R}^1_4 = e^{-2(\sigma+f)} \left[2\sigma_{14} - 2\sigma_1 \sigma_4 + 2\sigma_1 f' - 2\sigma_4 f' + \frac{f'}{r} \right],$$

where

$$f' \equiv \frac{df}{du}, \quad u \equiv r - t\ ; \quad \sigma_i = \frac{\partial \sigma}{\partial x^i}, \quad i = 1, 4. \qquad (3\cdot9)$$

The scalar curvature $\bar{R}$ is given by

$$\bar{R} = -6\, e^{-2(\sigma+f)} \left[\sigma_{11} - \sigma_{44} + \sigma_1{}^2 - \sigma_4{}^2 + \frac{\sigma_1}{r} \right]. \qquad (3\cdot10)$$

When $\bar{R} = 0$ the equations (3·7) and (3·10) are identical.

For an electromagnetic field we have from the equation (1·3)

$$\bar{R}^k_1 \ \bar{R}^4_k = 0$$

which is equivalent to

$$\bar{R}^4_1 \ (\bar{R}^1_1 + \bar{R}^4_4) = 0.$$

Here two cases arise :

Case 1

$$\bar{R}^4_1 = 0, \ \bar{R}^1_1 + \bar{R}^4_4 \neq 0.$$

Case 2

$$\bar{R}^4_1 \neq 0, \ \ \bar{R}^1_1 + \bar{R}^4_4 = 0.$$

Case 1 — $\bar{R}_1^4 = 0$. We have

$$\bar{R}_1^1 - \bar{R}_4^4 = 0.$$

and

$$\bar{R}_2^2 - \bar{R}_3^3 = 0.$$

These equations give

$$2\sigma_{14} - 2\sigma_1 \sigma_4 + 2\sigma_1 f' - 2\sigma_4 f' + \frac{f'}{r} = 0 ,$$

$$\sigma_{11} + \sigma_{44} + 2\sigma_4 f' - 2\sigma_1 f' - \sigma_1{}^2 - \sigma_4{}^2 - \frac{f'}{r} = 0,$$

$$\frac{\sigma_1}{r} = 0.$$

Thus we find that

$$\sigma_1 = 0,$$

$$2\sigma_4 = \frac{1}{r} ,$$

$$\sigma_{44} = \sigma_4{}^2$$

which is inconsistent.

Case 2— $\bar{R}_1^4 \neq 0$. In this case we also have

$$\bar{R}_1^1 + \bar{R}_4^4 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}11)$$

Also $\bar{R} = 0$ gives

$$\bar{R}_2^2 + \bar{R}_3^3 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (3{\cdot}12)$$

The equations (3·11) and (3·15) yield

$$2\sigma_{11} - 2\sigma_{44} + \sigma_1{}^2 - \sigma_4{}^2 + \frac{\sigma_1}{r} = 0,$$

$$\sigma_{11} - \sigma_{44} + 2\sigma_1{}^2 - 2\sigma_4{}^2 + \frac{2\sigma_1}{r} = 0.$$

The above two equations imply

$$\sigma_{11} - \sigma_{44} = 0 \text{ and } \sigma_1{}^2 - \sigma_4{}^2 + \frac{\sigma_1}{r} = 0 .$$

Therefore the solution for σ is

$$e^{2\sigma} = \frac{A}{(t-B)^2-r^2},$$

where A and B are arbitrary constants.

Also $\overline{R} = 0$ requires that

$$(\overline{R}^2_2)^2 = (\overline{R}^1_1)^2 - (\overline{R}^1_4)^2. \quad \text{.. .. ..} \quad (3{\cdot}14)$$

The equation (3·14) is not satisfied by (3·13). Hence the electromagnetic field conformal to the field (3·6) is not possible.

In order to satisfy the equation (3·14) by (3·13) we must have $f' = 0$. In this case the metric (3·8) becomes

$$ds^2 = \frac{A}{(t-B)^2-r^2}\left[(dt^2 - dr^2 - r^2d\phi^2 - dz^2)\right], \quad \text{..} \quad (3{\cdot}15)$$

which has already been obtained by Singh and Roy (1966) starting from flat metric using cylindrical polar coordinates. The electromagnetic behaviour of the line element (3·15) has been discussed by Singh and Roy (1966).

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