A NOTE ON A THEOREM OF PUTNAM AND WINTNER

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If A and B are operators on a Hilbert space H such that their commutator $C = AB - BA$ commutes with A and A has a logarithm which commutes with every operator that commutes with A, then it has been proved by Putnam and Wintner (*Proc. Am. math. Soc.*, **9** (1958), 360-62) that the spectrum of the operator $D = ABA^{-1} B^{-1}$ consists of 1 only. In this paper we have generalized the above result for elements of the form $A_1B - BA_2$, where A_1, A_2 and B are elements of a Banach algebra.

Let A, B, C be elements of a Banach algebra $\mathscr{B}$.

Then

$$C = AB - BA \qquad \ldots \qquad (1)$$

is called the commutator of A and B. Suppose that A commutes with C, i.e.

$$AC = CA \qquad \ldots \qquad (2)$$

then it has been proved by Kleinecke (1957) and Shirokov (1956) that

$$\text{sp}\ (C) = 0 \text{ only},$$

where sp(C) denotes the spectrum of C.

In case the Banach algebra $\mathscr{B}$ possesses the identity I and A and B are non-singular so that 0 does not belong to the spectrum of either A or B, then we can also consider the 'group' commutator defined by

$$D = ABA^{-1}\ B^{-1}.$$

Putnam and Wintner (1958) have given conditions under which (1) and (2) imply that

$$\text{sp}\ (D) = 1 \text{ only}.$$

The notion of a generalized commutator has been defined by Yadav and Ramanujan (1970). Let A_1, A_2 and B be elements of the Banach algebra $\mathscr{B}$. Then

$$C = A_1 B - BA_2 \qquad \ldots \qquad (3)$$

is called a generalized commutator.

Similarly if the Banach algebra $\mathscr{B}$ possesses the identity I and A_1, A_2, B are non-singular elements of $\mathscr{B}$, then

$$D = A_1 B A_2^{-1} B^{-1} \qquad \ldots \quad \ldots \quad \ldots \qquad (4)$$

is called a generalized group commutator.

Yadav and Ramanujan (1970) have studied conditions under which spectrum of D in (4) turns out to be consisting of 1 only. The result proved by them is a generalization of an earlier result due to Putnam. The aim of the present note is to generalize another result of Putnam and Wintner (1958) which is as follows:

Theorem 1—If A and B are operators on a Hilbert space H such that their commutator $C = AB - BA$ commutes with A and A has a logarithm which commutes with every operator that commutes with A, then the spectrum of the operator $D = AB\,A^{-1}\,B^{-1}$ consists of 1 alone.

We generalize the above result for elements of the form $C = A_1B - BA_2$ which satisfy the condition

$$A_1C = CA_2. \qquad \ldots \quad \ldots \quad \ldots \qquad (5)$$

We have the following :

Theorem 2—Let A_1, A_2 and B be non-singular elements of a Banach algebra $\mathscr{B}$ with identity I and suppose that $C = A_1B - BA_2$ satisfies the condition $A_1C = CA_2$. In addition, suppose that A_1 and A_2 have logarithms E_1 and E_2 respectively such that

$$A_1 = e^{E_1}, \; A_2 = e^{E_2} \; ; \; A_1 X = XA_2 \Rightarrow E_1 X = XE_2 \; (X \text{ arbitrary}). \qquad \ldots \qquad (6)$$

Then the spectrum of $D = A_1 BA_2^{-1} B^{-1}$ consists of 1 only.

In order to prove this theorem, we shall first prove the following:

Lemma—If A_1, A_2 and B are as assumed in Theorem 2 fins $C = A_1 B - BA_2$ satisfies the condition $A_1C = CA_2$, then

$$e^{tA_1}\, B\, e^{-tA_2} = B + tC \text{ for all real } t. \qquad \ldots \quad \ldots \quad \ldots \qquad (7)$$

PROOF : We have

$$e^{tA_1}\, B\, e^{-tA_2} = (J + tA_1 + \frac{t^2A_1^2}{2!} + \ldots.)\, B(I - tA_2 + \frac{t^2A_2^2}{2!} - \ldots.)$$

Put

$$S_n = I + tA_1 + \frac{t_1A_1^2}{2!} + \ldots. + \frac{t_nA_1^n}{n!}.$$

and

$$T_n = I - tA_2 + \frac{t^2A_2^2}{2!} \ldots. + (-1)^n \frac{t^n A_2^n}{n!}$$

Then,

$$S_n B T_n = \left(I + tA_1 + \frac{t^2 A_1^2}{2!} + \ldots + \frac{t^n A_1^n}{n!} \right)$$

$$B \left(I - tA_2 + \frac{t^2 A_2^2}{2!} - \ldots + (-1)^n \frac{t^n A_2^n}{n!} \right)$$

$$= \sum_{i=0}^{n} \frac{1}{i\,!} t^i D_i \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8)$$

where

$$D_0 = B$$

and

$$D_i = A_1^i B - iA_1^{i-1} BA_2 + \frac{i(i-1)}{2!} A_1^{i-2} BA_2^2 - \ldots + (-1)^i BA_2^i,$$

$$i = 1, 2, \ldots, n.$$

One can show that $D_i = 0$ for $i > 1$.

Indeed,

$$D_i = A_1^{i-1} (A_1 B - BA_2) - (i-1) A_1^{i-2} (A_1 B - BA_2) A_2 + \frac{(i-1)(i-2)}{2!}$$

$$A_1^{i-3} (A_1 B - BA_2) A_2^2 - \ldots + (-1)^{i-1} (A_1 B - BA_2) A^{i-1}.$$

$$= A_1^{i-1} C - (i-1) A_1^{i-1} CA_2 + \frac{(i-1(i-2)}{2!} A_1^{i-3}$$

$$CA_2^2 - \ldots + (-1)^{i-1} CA_2^{i-1}.$$

$$= A_1^{i-1} C \left\{ (1 - (i-1) + \frac{(i-1)(i-2)}{2!} - \ldots + (-1)^{i-1} \right\}.$$

$$= A^{i-1} C \left\{ (1-1)^{i-1} \right\}$$

$$= 0.$$

Hence (8) reduces to

$$S_n B T_n = B + tC.$$

Taking limit as $n \to \infty$, we get (7), and this completes the proof of the lemma.

PROOF OF THEOREM 2: We see that $(I + tCB^{-1})\, B = B + tC = e^{tA_1}\, B\, e^{-tA_2}$ is invertible for all t.

Hence

$$\text{sp}\,(CB^{-1}) = 0 \text{ only.}$$

Now, we define F by

$$F = E_1\, B - BE_2. \qquad (9)$$

Since $A_1E_1 = E_1A_1$ and $A_2E_2 = E_2A_2$, it follows that

$$A_1F - FA_2 = E_1C - CE_2.$$

Now $A_1C = CA_2$ implies $E_1C = CE_2$ by (6). This gives

$A_1F = FA_2$. Again by (6) we have

$$F_1F = FE_2. \qquad (10)$$

Now as argued above, with the help of (9) and (10), we have

$$\text{sp}\,(FB^{-1}) = 0 \text{ only}$$

and $e^{tE_1}\, B\, e^{-tE_2} = B + tF$ for all real t.

Putting $t = 1$, we get

$$e^{E_1}\, B\, e^{-E_2} = B + F.$$

Now,

$$\begin{aligned} A_1B - BA_2 &= e^{E_1}\, B - Be^{E_2} \\ &= (e^{B_1}\, B\, e^{-E_2} - B)\, e^{E_2} \\ &= F\, e^{E_2} \\ &= F\, A_2 \end{aligned}$$

i.e. $\quad C = F\, A_2,$

Since $\text{sp}\,(FB^{-1}) = 0$ only implies $\text{sp}\,(CA_2^{-1}\, B^{-1}) = 0$ only, the theorem follows.

In the end we remark that Theorem 2 of Putnam and Wintner (1968) can similarly be generalized. Also one can construct an algebraic proof of our Theorem 2 on the lines of Herstein (1958).

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References

Herstein, I. N. (1958). On a theorem of Putnam and Wintner. *Proc. Am. math. Soc.*, **9**, 363-64.

Kleinecke, D. C. (1957). On operator commutators. *Proc. Am. math. Soc.* **8**, 535-536.

Putnam, C. R. On the Spectra of group commutators. *Bell Un. Mat. Ital.*, Series III, Anno. XV, No. 3, 379-383.

Putnam, C. R. and Wintner, A. (1958). On the spectra of group commutators. *Proc. Am. math. Soc.*, **9**, 360-62.

Shirokov, F. V. (1956). Proof of a conjecture of Kaplansky. *Uspehi Math. Nauk*, **11**, 167.

Yadav, B. S. and Ramanujan, P. B. (1970). On commutators of operators. *Math. Japon.*, **15**, No. 1.