SOME FINITE SUMMATIONS INVOLVING *H*-FUNCTION I

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Sharma and Abiodun (1971) have obtained some finite summations involving the G function with the help of an integral given by Shah (1969). The author has earlier also obtained several similar results using the different known identities In this paper some of these results are generalized to obtain some finite summations involving the H-function of one and two variables, using the same technique as that applied for deducing the results for the G-function.

1. In a recent paper Sharma and Abiodun (1971) have obtained some finite summations involving Meijer's G-function with the help of an integral given by Shah (1969). Manglik (*in press*) and Agrawal and Manglik (*in press*) have also obtained several similar results using the different identities (Agrawal and Manglik *in press*). The natural generalization of G-function is Fox's'H-function and Sharma and Abiodun (1971) have not given any results involving H- functions. Probably, this may be due to the fact that an integral similar to Shah (1969). which is the main tool in the derivation of their results for G-functions is not available for H-functions. On the other hand, the results of Manglik (*in press*) and Agrawal and Manglik (*in press*) can be generalized in a very simple way, using the same technique as that applied for deducing the results for G-functions. Accordingly, in this paper we shall obtain some finite summations involving H-functions of one and two variables and product of two H-functions.

2. We shall use the result [6, (4·1)] :

$$(1+a)_2\; F_1\left[\begin{matrix} d-b,\, d-a-1 \\ d+1 \end{matrix}\right]_n - (1+b)_2\, F_1\left[\begin{matrix} d-b-1,\, d-a \\ d+1 \end{matrix}\right]_n = \frac{(a-b)}{n\,!}\;\frac{(d-b)_n(d-a)_n}{(d+1)_n} \quad \ldots \quad \ldots \qquad (2\cdot1)$$

where the suffix n on the left side indfcates that only first n terms of the F series are to be included in the expansion.

The *H*-function introduced by Fox (1971) will be represented and defined as follows

$$\mathrm{H}_{p,q}^{l,u}\left(z\left|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right.\right) = \mathrm{H}_{p,q}^{l,u}\left(z\left|\begin{matrix}(a_1, e_1), \ldots\ldots, (a_p, e_p)\\(b_1, f_1), \ldots\ldots, (b_q, f_q)\end{matrix}\right.\right)$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l} \Gamma(b_j - f_j s) \prod_{j=1}^{u} \Gamma(1 - a_j + e_j s)}{\prod_{j=l+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=u+1}^{p} (a_j - e_j s)} z^s\, ds = \frac{1}{2\pi i}\int_L f(s)\, z^s\, ds \quad (2\cdot2)$$

where an empty product is interpreted as unity, $0 \leqslant l \leqslant q$, $0 \leqslant u \leqslant p$; e's and f's are all positive; L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j s)$, $j=1,\ldots,..l$ life on the right side of the contour and those of $\Gamma(1-a_j+e_j s)$ $j=1,..u$ lie on the left side of the contour.

Recently, Braaksma (1963) has discussed, asymptotic expansions and analytic continuations for the *H*-function.

3. The following results will be proved in this section:

$$\sum_{r=0}^{n} \frac{1}{r!\,\Gamma(1+d+r)}\left[\mathrm{H}\left(z\left|\begin{matrix}(-1-a, e), (a_p, e_p), (d-b, f), (d-a-1, e)\\(d-b+r, f), (d-a+r-1, e), (b_q, f_q), (-a, e)\end{matrix}\right.\right)\right.$$

$$\left. - \mathrm{H}\left(z\left|\begin{matrix}(-1-b, f), (a_p, e_p), (d-b-1, f), (d-a, e)\\(d-b+r-1, f), (d-a+r, e), (b_q, f_q), (-b, f)\end{matrix}\right.\right)\right]$$

$$= \frac{1}{n!\,\Gamma(1+d+n)}\left[\mathrm{H}\left(z\left|\begin{matrix}(-a, e), (a_p, e_p), (d-b, f), (d-a, e)\\(d-b+n, f), (d-a+n, e), (b_q, f_q), (1-a, e)\end{matrix}\right.\right)\right.$$

$$\left. - \mathrm{H}\left(z\left|\begin{matrix}(-b, f), (a_q, e_p), (d-b, f), (d-a, e)\\(d-b+n, f), (d-a+n, e), (b_q, f_q), (1-b, f)\end{matrix}\right.\right)\right] \quad .. \quad (3\cdot1)$$

$$\sum_{r=0}^{n} \frac{1}{r!}\left[(1+a).\, H\left(z\left|\begin{matrix}(a_p, e_p), (d-b, h), (d-a-1, h), (d+1+r, h)\\(d-b+r, h), (d-a-1+r, h), (b_q, f_q)\end{matrix}\right.\right)\right.$$

$$\left. - (1+b)\, H\left(z\left|\begin{matrix}(a_p, e_p), (d-b-1, h), (d-a, h), (d+1+r, h)\\(d-b-1+r, h), (d-a+r, h), (b_q, f_q)\end{matrix}\right.\right)\right]$$

$$= \frac{a-b}{!n}\mathrm{H}\left(z\left|\begin{matrix}(a_p, e_p), (d-b, h), (d-a, h), (d+1+n, h)\\(d-b+n, h), (d-a+n, h), (b_q, f_q)\end{matrix}\right.\right) \quad .. \quad (3\cdot2)$$

$$\sum_{r=0}^{n} \frac{1}{r!}\Big[\frac{(d-b)_a}{(d+1)_r} H\left(z \,\Big|\, \begin{matrix} (-1-a, e), (a_p, e_p), (d-a-1, e) \\ (d-a-1+r, e), (b_q, f_q), (-a, e) \end{matrix}\right)$$

$$- (1+b)\frac{(d-b-1)_r}{(d+1)_r} H\left(z \,\Big|\, \begin{matrix} (a_p, e_p), (d-a, e) \\ (d-a+r, e), (b_q, f_q) \end{matrix}\right)\Big]$$

$$= \frac{(d-b)_n}{(d+1)_n\, n!} H\left(z \,\Big|\, \begin{matrix} (b-a, e), (a_p, e_p), (d-a, e) \\ (d-a+n, e), (b_q, f_q), (1+b-a, e) \end{matrix}\right). \quad \cdot\cdot \quad (3{\cdot}3)$$

Conditions for the validity of the above results and the suffix of H have been omitted as no ambiguity arises without them and they are obvious.

4. To establish (3.1), employ (2-2) on the left-hand side of (3.1). We get

$$\sum_{r=0}^{n} \frac{1}{r!}\frac{1}{\Gamma(1+d+r)} \cdot \frac{1}{2\pi i}\int_L f(s)\Big[\frac{\Gamma(2+a+es)\,\Gamma(d-b+r-fs)\,\Gamma(d-a-1+r\text{-}es)}{\Gamma(1+a+es)\;\Gamma(d-b-fs)\;\Gamma(d-a-1-es)}$$

$$-\frac{\Gamma(2+b+fs)\;\Gamma(d-b-1+r-fs)\;\Gamma(d-a+r-es}{(1+b+fs)\;\Gamma(d-b-1-fs)\;\Gamma(d-a-es)}\Big] z^s\, \mathrm{ds}.$$

Justifiably, changing the order of summation (finite) and integration, we get,

$$\frac{1}{2\pi i}\int_L f(s)\sum_{r=0}^{u}\frac{1}{(1+d+r)\,r!}\;[\,(1+a+es)\,(d-b-fs)_r\,(d-a-1-es)_r$$

$$-(1+b+fs)\,(d-b-1-fs)_r\,(d-a-es)_r]\; z^s\, ds$$

Now using (2·1) we get the right hand side of (3·1) which proves the result. Similarly we can prove (3·2) and (3·3). In fact, not only the three results given in section 3 above but some more results can be obtained from (2·1).

5. Now, we give a very elegant result involving the product of two H-functions. The result can be proved in the same way as that of proving the results of section 3, using (2·1) and (2·2).

$$\sum_{r=0}^{n}\frac{1}{r!\,\Gamma(1+d+r)}\Big[H\left(x \,\Big|\, \begin{matrix} (-1-a,e), (a_p, e_p), (d-a-1,e) \\ (d-a-1+r,e), (b_q, f_q), (-a,e) \end{matrix}\right) \times$$

$$H\left(y \,\Big|\, \begin{matrix} (a_p, e_p), (d-b, f) \\ (d-f+r, f), (b_q f_q) \end{matrix}\right) -$$

$$H\left(x \,\Big|\, \begin{matrix} (a_p, e_p), (d-a, e) \\ (d-a+r,e), (b_q, f_q) \end{matrix}\right) H\left(y \,\Big|\, \begin{matrix} (-1-b,f), (a_p, e_p), (d-b-1,f) \\ (d-b-1+r,f), (b_q, f_q), (-b,f) \end{matrix}\right)\Big]$$

$$= \frac{1}{n!\ \Gamma(1+d+n)} \Bigg[H\left(x \,\middle|\, \begin{matrix} (-a,e),\ (a_p, e_y),\ (d-a,e) \\ (d-a<n,e),\ (b_q, f_q),\ (1-a,e) \end{matrix} \right) \times H\left(y \,\middle|\, \begin{matrix} (a_p, e_p),\ (d-b, f) \\ (d-b+n, f),\ (b_q, f_q) \end{matrix} \right) -$$

$$H\left(x \,\middle|\, \begin{matrix} (a_p, e_p),\ (d-a,e) \\ (d-a+n,e),\ (b_q, f_q) \end{matrix} \right) H\left(y \,\middle|\, \begin{matrix} (-b, f),\ (a_p, e_p),\ (d-b, f) \\ (d-b+n, f),\ (b_q, f_q),\ (1-b, f) \end{matrix} \right) \Bigg] \quad \text{.. (5·1)}$$

6. Recently, Agarwal and Mathur (1969, p. 536) have introduced the H-function of two variables in the form of Mellin-Barnes type integral. This will be represented and defined as follows :

$$H\begin{bmatrix} x \\ y \end{bmatrix} \equiv H\left[x, y \,\middle|\, \begin{bmatrix} m_1, o \\ p_1\ q_1 \end{bmatrix} \begin{matrix} (a_{p_1}, \alpha_{p_1}) \\ (b_{q_1}, \beta_{q_1}) \end{matrix} \,\middle|\, \begin{pmatrix} n_2, m_2 \\ p_2, q_2 \end{pmatrix} \begin{matrix} (c_{p_2}, \gamma_{p_2}) \\ (d_{q_2}\ \delta_{q_2}) \end{matrix} \,\middle|\, \begin{pmatrix} n_3, m_3 \\ p_3, q_3 \end{pmatrix} \times \begin{matrix} (e_{p_3}, \sigma_{p_3}) \\ (f_{q_3}\ \epsilon_{q_3}) \end{matrix} \right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L'} \int_{L''} \phi(s+t)\ \psi(s, t)\ x^s\ y^t\ ds\ dt. \quad \text{.. .. .. (6·1)}$$

where (i) (a_{p_1}, α_{p_1}) represents the set of parameters (a_1, α_1), $(a_2, \alpha_2), \ldots (a_{m_1}, \alpha_{m_1})$; $(a_{m_1+1}, \alpha_{m_1+1}) \ldots (a_{p_1}, \alpha_{p_1})$. Similar sets of parameters are represented by (b_{q_1}, β_{q_1}) (c_{p_2}, γ_{p_2}), (d_{q_2}, δ_{q_2}) etc..

(ii) α, β, γ, δ, σ and ϵ are all positive;

(iii) L' and L'' are suitable contours.

(iv) $$\phi(s+t) = \frac{\prod_{j=1}^{m_1} \Gamma(a_j + \alpha_j s + \alpha_j t)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - \alpha_j s - \alpha_j t) \prod_{j=0}^{q_1} \Gamma(b_j + \beta_j s + \beta_j t)},$$

$$\psi(s, t) = \frac{\prod_{j=1}^{m_2} \Gamma(1-c_j+\gamma_j s) \prod_{j=1}^{n_2} \Gamma(d_j-\delta_j s) \prod_{j=1}^{m_3} \Gamma(1-e_j+\sigma_j t(\prod_{j=1}^{q_3} \Gamma(f_j+\epsilon_j t)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j-\gamma_j s) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+\delta_j s) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-\sigma_j t) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+\epsilon_j t)}.$$

If $2(m_1 + m_2 + n_2) > p_1 + q_1 + p_2 + q_2$

$2(m_1 + m_3 + n_3) > p_1 + q_1 + p_3 + q_3$

$|\arg(x)| < [m_1 + m_2 + n_2 - \frac{1}{2}(p_1 + q_1 + p_2 + q_2)]\,\pi$

$|\arg(y)| < [m_1 + m_3 + n_3 - \frac{1}{2}(p_1 + q_1 + p_3 + q_3)]\,\pi$.

or $[p_1 + p_2 < q_1 + q_2, p_1 + p_3 < q_1 + q_3]$ or $[p_1 + p_2 = q_1 + q_2, p_1 + p_3 = q_1 + q_3]$ and $|x| < 1$, $|y| < 1$, the double integral (6·1) will be convergent.

7. In this section we prove a result involving H-function of two variables defined in section 6.

$$\sum_{r=0}^{n} \frac{1}{r!} \left\{ \mathbf{H} \left[y, x \left| \begin{bmatrix} m_1+1, 0 \\ p_1+1, q_1+1 \end{bmatrix} \begin{matrix} (d-a+r-1, h), (a_{p_1}, \alpha_{p_1}) \\ (b_{q_1}, \beta_{q_1}), (d-a-1, h) \end{matrix} \right| \begin{pmatrix} n_2, m_2+1 \\ p_2+1, q_1+1 \end{pmatrix} \right. \right.$$

$$\times \left. \begin{matrix} (1-d+b-r, h), (c_{p_2}, \gamma_{p_2}) \\ (d_{q_2}, \delta_{q_2}), (1-d+b, h), (-d-r, h) \end{matrix} \right| \begin{pmatrix} n_3+1, m_3 \\ p_3+1, q_3+1 \end{pmatrix} \left. \begin{matrix} (e_{p_3}, \sigma_{p_3}), (1+a, h \\ (2+a, h), (f_{q_3}, \epsilon_{q_3}) \end{matrix} \right| \Bigg]$$

$$-(1+b), \mathbf{H} \; x, y \left| \begin{bmatrix} m_1+1, 0 \\ p_1+1, q_1+1 \end{bmatrix} \begin{matrix} (d-a+r, h), (a_{p_1}, \alpha_{p_1}) \\ (b_{q_1}, \beta_{q_1}), (d-a, h) \end{matrix} \right| \begin{pmatrix} n_2, m_2+1 \\ p_2+1, q_2+1 \end{pmatrix}$$

$$\times \left. \begin{matrix} (2-d+b-r, h), (c_{p_2}, \gamma_{p_2}) \\ (d_{q_2}, \delta_{q_2}), (2-d+b, h), (-d-r, h) \end{matrix} \right| \begin{pmatrix} n_3 \; m_3 \\ p_3, q_3 \end{pmatrix} \left. \begin{matrix} (e_{p_3}, \sigma_{p_3}) \\ (f_{q_3}, \epsilon_{q_3}) \end{matrix} \right| \Bigg] \Bigg\}$$

$$= \frac{1}{n!} \mathbf{H} \left[x, y \left| \begin{bmatrix} m_1+1, 0 \\ p_1+, q_1+1 \end{bmatrix} \begin{matrix} (d-a+n, h), (a_{p_1}, \alpha_{p_1}) \\ (b_{q_1}, \beta_{q_1}), (d-a, h) \end{matrix} \right| \begin{pmatrix} n_2, m_2+1 \\ p_2+1, q_2+1 \end{pmatrix} \right.$$

$$\times \left. \begin{matrix} (1-d+b-n, h), (c_{p_2}, \gamma_{p_2}) \\ (d_{p_2}, f_{q_2}), (1-d+b, h), (-d-n, h) \end{matrix} \right| \begin{pmatrix} (n_3+1, m_3 \\ p_3+1, q_3+1 \end{pmatrix} \left. \begin{matrix} (e_{p_3}, \sigma_{p_3}), (a-b, h) \\ (a-b+1, h), (f_{q_3}, \epsilon_{q_3}) \end{matrix} \right| \Bigg] \qquad \ldots \quad (7{\cdot}1)$$

where the conditions for the validity are obvious from (6·2).

PROOF: To establish (7·1), apply (6·1) on its left hand side. We get,

$$\sum_{r=0}^{n} \frac{1}{r!} \frac{1}{(1+d+r+hs)} \cdot \frac{1}{(2\pi i)^2} \int_{L'} \int_{L''} \phi(s+t) \, \psi(s, t).$$

$$\left[\frac{\Gamma(2+a-ht) \, \Gamma(d-b+hs+r) \, \Gamma(d-a+hs+ht-1+r)}{\Gamma(1+a-ht) \, \Gamma(d-b+hs) \, \Gamma(d-a+hs+ht-1)} - (1+b) \right.$$

$$\left. \times \frac{\Gamma(d-b-1+hs+r)}{\Gamma(d-b..1+hs)} \cdot \frac{\Gamma(d-a+hs+ht+r)}{\Gamma(d-a+hs+ht)} \right] x^s y^t \, ds \, dt.$$

Now changing the order of summation (finite and integration, we get,

$$\frac{1}{(2\pi i)^2} \int_{L'} \int_{L''} \phi(s+t) \, \psi(s, t). \sum_{r=0}^{n} \frac{1}{r! \, \Gamma(1+d+r+hs)} \cdot [(1+a-ht)\,(d-b+hs)_r$$

$$\times (d-a+hs+ht_{-1})_r - (1+b)\,(d-b-1+hs)_r\,(d-a+hs+ht)_r] \, . \; x^s y^t \, ds \, dt.$$

Now using (2·1) we get the right-hand side of (7·1). This proves the result.

8. Putting all e's $= 1$ and f's $= 1$ and h's $= 1$ in the results of sections 3 and 5 we get different results of (Manghik 1974). If we put $p_1 = q_1 = m_1 = 0$ in (7·1) and intesesting result involving product of two H-functions is obtained.

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